Directly Estimating Nonclassicality

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We establish a method of directly measuring and estimating nonclassicality—operationally defined in terms of the distinguishability of a given state from one with a positive Wigner function. It allows us to certify nonclassicality, based on possibly much fewer measurement settings than necessary for obtaining complete tomographic knowledge, and is at the same time equipped with a full certificate. We find that even from measuring two conjugate variables alone, one may infer the nonclassicality of quantum mechanical modes. This method also provides a practical tool to eventually certify such features in mechanical degrees of freedom in opto-mechanics. The proof of the result is based on Bochner's theorem characterizing classical and quantum characteristic functions and on semidefinite programming. In this joint theoretical-experimental work we present data from experimental optical Fock state preparation.

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Where is the "boundary" between classical and quantum physics? Unsurprisingly, acknowledging that quantum mechanics is the fundamental theory from which classical properties should emerge in one way or the other, instances of this question have a long tradition in physics. Possibly the most conservative and stringent criterion for nonclassicality of a quantum state of bosonic modes is that the Wigner function—the closest analogue to a classical probability distribution in phase space—is negative, and can hence no longer be interpreted as a classical probability distribution [1–3]. From this, negativity of other quasi-probability distributions, familiar in quantum optics, such as the P-function [1,4] follows. In fact, a lot of experimental progress was made in recent years on preparing quantum states of light modes that exhibit such nonclassical features, when preparing number states, photon subtracted states, or small Schrödinger cat states [5–8]. At the same time, a lot of effort is being made of driving mesoscopic mechanical degrees of freedom into quantum states eventually showing such nonclassical features [9]. All this poses the question, needless to say, of how to best and most accurately certify and measure such features.

In this work, (i) we demonstrate that, quite remarkably, nonclassicality in the above sense can be detected from mere measurements of two conjugate variables. For a single mode, this amounts to position and momentum detection, as can be routinely done by homodyne measurements in optical systems. (ii) What is more, using such data (or also data that are tomographically complete) one can get a direct and rigorous lower bound to the probability of operationally distinguishing this quantum state from one with a positive Wigner function—including a full certificate. Such a bound uses information from possibly much fewer measurement settings than needed for full quantum state tomography. At the same time, quantum state tomography using Radon transforms for quantum modes is overburdened with problems of ill-conditioning.

The method introduced here, in contrast, is a *direct method* giving rise to a *certified bound* which arises from conditions all classical and quantum characteristic functions have to satisfy as being grasped by the classical and quantum Bochner's theorem [10]. Hence, we ask: "What is the smallest nonclassicality consistent with the data"? Intuitively speaking, the proof circles around the deviation of a quantum characteristic function as the Fourier transform of the Wigner function from a classical characteristic function. This deviation can then be formulated in terms of a semidefinite program—so a well-behaved convex optimization problem—giving rise to certifiable bounds. The same technique can also be applied to notions of entanglement, and indeed, the rigor applied here reminds of applying quantitative entanglement witnesses [11,12]. What is more, the criterion evaluation procedure is efficient. At present, such techniques should be most applicable to systems in quantum optics, and we indeed implement this idea in a quantum optical experiment preparing a field mode in a nonclassical state. Yet, they should be expected to be helpful when eventually certifying that a mesoscopic mechanical system has eventually reached quantum properties [9], where "having achieved a nonclassical state", with careful error analysis, will constitute an important benchmark.

Measure of nonclassicality.—Nonclassicality is most reasonably quantified in terms of the possibility of operationally distinguishing a given state from a state that one would conceive as being classical. That is to say, the meaningful notion of distinguishing a state from a classical one is as follows.

Definition 1: (Measure of nonclassicality).—Nonclassicality is measured in terms of the operational distinguishability of a given state from a state having a positive Wigner function,

$$\eta(\rho) = \min_{\omega \in \mathcal{C}} \|\rho - \omega\|_1, \tag{1}$$

where $\mathcal{C}$ denotes the set of all quantum states with positive Wigner function and $\| \cdot \|_1$ is the trace norm.

This measure is indeed *the* operational definition of a nonclassical state—as long as one accepts the negativity of the Wigner function as the figure of merit of nonclassicality. Needless to say, the operational distinguishability with respect to other properties would also be quantified by trace-distances, and naturally several quantities of such a type can be found in the literature (see, e.g., Ref. [13] for a similar notion of nonclassicality and Ref. [14] for a related idea to quantify entanglement). It has the following natural properties: It is (a) invariant under passive and active linear transformations, and (b) nonincreasing under Gaussian channels, and in fact under any operation that cannot map a state with a positive Wigner function onto a negative one. The latter property is an immediate consequence of the trace norm being contractive under completely positive maps. Moreover, since Gaussian states are positive this measure of negativity gives a direct lower bound to the non-Gaussianity of the same state—quantified again in terms of the distance to the set of Gaussian states. Such a notion of non-Gaussianity (see, e.g., Refs. [15]), just as the negativity of the Wigner function as such, can be viewed as quantifying a *resource* in quantum information processing. Similar to entanglement measures being monotones under local operations with classical communication, these measures are monotones under Gaussian operations. What is more, the negativity of the Wigner function may also be seen as quantifying the potential of violating a Bell inequality based on homodyning [16].

Characteristic functions and Bochner's theorems.—We consider physical systems of n bosonic modes, associated with canonical coordinates $R = (q_1, \ldots, q_n, p_1, \ldots, p_n)$, of "position" and "momentum", or some quadratures. In the center of the analysis will be quantum characteristic functions [2,17], for n modes as a function χ: $\mathbb{R}^{2n} \to \mathbb{C}$ defined as $\chi(\xi) = \mathrm{tr}[\rho D(\xi)]$, $D(\xi) = e^{i\xi\cdot\sigma R}$, so as the expectation value of the *Weyl* or *displacement operator* [18]. This characteristic function is nothing but the Fourier transform of the familiar *Wigner function* W: $\mathbb{R}^{2n} \to \mathbb{R}$,

$$W(z) = \frac{1}{(2\pi)^{2n}} \int \chi(\xi) e^{-i\xi\cdot\sigma z} d\xi. \tag{2}$$

A key tool will be the notion of λ *positivity* [10]:

Definition 2: (λ positivity).—A function χ: $\mathbb{R}^{2n} \to \mathbb{C}$ is λ-positive definite for $\lambda \in \mathbb{R}$ if for every $m \in \mathbb{N}$ and for every set of real vectors $T = (\xi_1, \xi_2, \ldots, \xi_m)$ the $m \times m$ matrix $M^{(\lambda)}(\chi, T)$ is non-negative, $M^{(\lambda)}(\chi, T) \geq 0$, with

$$(M^{(\lambda)}(\chi, T))_{k,l} = \chi(\xi_k - \xi_l) e^{i\lambda\xi_k\cdot\sigma\xi_l/2}. \tag{3}$$

Conversely, one can ask for a classification of all functions that can be characteristic functions of a quantum state, or some classical probability distribution. Such a characterization is captured in the quantum and classical Bochner's theorems [10]. (i) Every characteristic function of a quantum state must be *1-positive definite*. (ii) Every characteristic function of a quantum state with a positive Wigner function must be at the same time *1-positive definite* and *0-positive definite*.

Measuring nonclassicality.—Data are naturally taken as slices in phase space, resulting from measurements of some linear combinations of the canonical coordinates, as they would be obtained from a phase sensitive measurement such as homodyning in quantum optics. One collects data from measuring observables $u_k R$ for some collection of $u_k \in \mathbb{R}^{2n}$ with $\|u_k\| = 1$. E.g., in the simplest case of one mode one could measure only q and p or, if the state is phase invariant, one could average over all the possible directions. With repeated measurements one can estimate the associated probability distributions P_k: $\mathbb{R} \to \mathbb{R}^+$, related to slices of the characteristic functions by a simple Fourier transform $\int P_k(s) e^{i\omega s} ds = \chi(\omega\sigma u_k)$. Actually, in a real experiment one can build only a statistical histogram rather than a continuous probability distribution. Hence, measurements of values of the characteristic function must be equipped with error bars [19]

$$\delta(\omega) = |\omega| h + n/\sqrt{N}, \tag{4}$$

where $2h$ is the width of each bin of the histogram, N is the number of measurements and n is the number of standard deviations that one should consider depending on the desired level of confidence [20]. This kind of measurements can be performed also in *optomechanical systems* where a particular quadrature of a mechanical oscillator can be measured *a posteriori* by appropriately homodyning a light mode coupled to it [21]. A different idea has recently been proposed for directly pointwise measuring the characteristic function of a mechanical mode coupled to a two-level system [22]. In both cases the method that we are going to describe can be easily applied. Restricted measurements also arise in the context of *bright beams* [23], where Mach-Zehnder interferometers have to replace homodyning. In the study of states of *macroscopic atomic ensembles* [24] similar issues arise.

Bounds to the nonclassicality from convex optimization.—We assume that we estimate the values of the characteristic function $\chi(\bar{\xi}_j) \simeq c_j$ for a given set of points $\bar{\xi}_j$, $j = 1, \ldots, p$, within a given error $\delta_j \geq 0$ [20], so that $|\chi(\bar{\xi}_j) - c_j| \leq \delta_j$. Now pick a set of suitable test vectors $T = (\xi_1, \ldots, \xi_m)$, the differences $\xi_j - \xi_k$ of which at least contain the data points $\bar{\xi}_1, \ldots, \bar{\xi}_p$. Based on this, we define the following convex optimization problem as over χ, x,

$$\text{minimize } x, \tag{5}$$

such that

$$|\mathrm{Re}(\chi(\bar{\xi}_j)) - \mathrm{Re}(c_j)| \leq \delta_j, \qquad j = 1, \ldots, p, \tag{6}$$

$$|\mathrm{Im}(\chi(\bar{\xi}_j)) - \mathrm{Im}(c_j)| \le \delta_j, \qquad j = 1, \ldots, p, \tag{7}$$

$$M^{(0)}(\chi, T) + xm\mathbb{1} \ge 0, \qquad M^{(1)}(\chi, T) \ge 0, \tag{8}$$

where $M^{(0)}(\chi, T)$ and $M^{(1)}(\chi, T)$ are the Hermitian matrices (3) associated with the λ positivity, based on the test points $\xi_1, \ldots, \xi_m$ as being specified in Def. 2. The minimization is in principle performed over all functions χ: $\mathbb{R}^{2n} \to \mathbb{C}$ such that $\chi(-\xi) = \chi(\xi)^*$, where $\chi(\bar{\xi}_l)$ is constrained by the data and $M^{(0/1)}(\chi, T)$ depend on the test points. Since we take only a finite number of points of χ, yet, the above problem gives rise to a *semidefinite problem (SDP)* [25]. This can be efficiently solved with standard numerical algorithms. By means of the notion of Lagrange duality, one readily gives *analytical certifiable bounds*: Every solution for the dual problem will give a proven lower bound to the primal problem [25], and hence a lower bound to the measure of nonclassicality itself. The entire procedure hence amounts to an arbitrarily tight *convex relaxation* of the Bochner constraints. We can now formulate the main result: Eq. (5) gives rise to a lower bound for the nonclassicality: Given the data (and errors), one can find good and robust bounds to the smallest nonclassicality that is consistent with the data.

Theorem 3: (Estimating nonclassicality).—The output x' of Eq. (5) is a lower bound for the nonclassicality, $\eta(\rho) \ge x'$. The proof proceeds by constructing a *witness operator*

$$F = \frac{1}{m} \sum_{k,l=1}^{m} v_k^* v_l D(\xi_k - \xi_l), \tag{9}$$

where $\xi_1, \ldots, \xi_m \in \mathbb{R}^{2n}$ are the test vectors from Bochner's theorem used in the SDP and v is the normalized eigenvector associated with the minimum eigenvalue of $M^{(0)}(\chi', T)$, where χ' is the optimal solution for χ. For a given state ρ, this operator F has the following properties: (i) $F = F^\dagger$, (ii) $|\mathrm{tr}(F\omega)| \le 1$ for all quantum states ω. (iii) $\mathrm{tr}(F\omega) \ge 0$ for all quantum states $\omega \in \mathcal{C}$. (4) If $x' \ge 0$ is the optimal solution, then $\mathrm{tr}(F\rho) \le -x'$. These properties will be proven to be valid in the Supplementary material [19], involving some technicalities. They suggest that F is actually a *witness observable* able to distinguish a subset of nonclassical states from the convex set of classical states. Formally, from the variational definition of the trace norm, we have the lower bound to be shown [26],

$$\eta(\rho) = \min_{\omega \in \mathcal{C}} \|\rho - \omega\|_1 \ge \min_{\omega \in \mathcal{C}} \mathrm{tr}(\omega F) - \mathrm{tr}(F\rho) \ge x'. \tag{10}$$

An example: Schrödinger cat state.—As an example we consider a quantum superposition of two coherent states, so $|\psi\rangle \sim (|\alpha\rangle + |-\alpha\rangle)$ with $\alpha = 1.77$. We assume to measure only the probability distributions of position and momentum [Fig. 1(a)]: $P_1(q) = |\langle q|\psi\rangle|^2$ and $P_2(p) = |\langle p|\psi\rangle|^2$, i.e., the data is collected from a mere *pair of canonical operators*. This amount of information is of course not sufficient for tomographically reconstructing the state since it corresponds to just two orthogonal slices of the characteristic function. In order to define the SDP we consider a 25×25 square lattice centered at the origin of the domain of the characteristic function, optimizing over the values of χ at the lattice points. Position and momentum measurements define the constraints (6) and (7) for only two slices of the lattice (assuming an error of $\delta_j = 10^{-3}$ for each point). We generate 100 random test vectors and we construct the associated λ-positivity constraints (8). The output of the SDP is $x' \simeq 0.05 > 0$ which is a certified lower bound for the nonclassicality of the state.

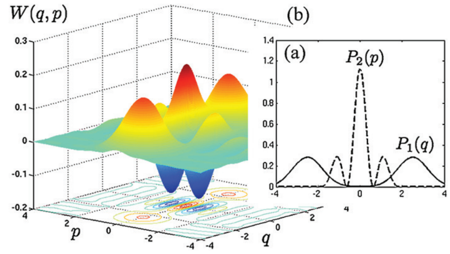

FIG. 1 (color online). (a) Position and momentum distributions for an exact cat state. (b) Wigner function based on the SDP.

Experimentally detecting nonclassicality.—Finally, to certify the functioning of the idea in a quantum optical context, we apply our method to experimental data. We consider data from a heralded single-photon source based on parametric down-conversion (cf. Ref. [8]). Here, an optical parametric oscillator (OPO), pumped continuous-wave and far below threshold, delivers the down-converted photon pair at frequencies $\omega_\pm$. The pair is separated using an optical cavity; the transmitted photon ω_- is frequency filtered by additional cavities before impinging an avalanche photodiode (APD) giving the heralding event for homodyne measurement of the reflected twin photon ω_+. On every event the homodyne current is sampled around the heralding time and postprocessed into one quadrature value using an appropriate mode function [27]. In total, quadrature values from 180 000 events are accumulated [Fig. 2(a)]. Data is phase randomized meaning that we can use the same probability distribution for every phase space direction, the phase being unavailable in the experiment. Since our nonclassicality measure is convex, averaging over phase space directions is an operation which can only decrease the negativity of the state. This means

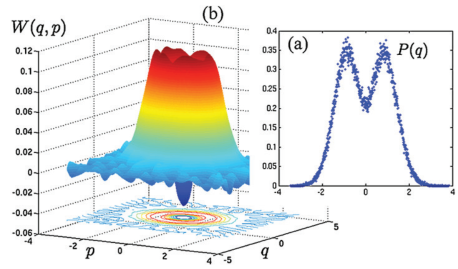

FIG. 2 (color online). (a) Raw measured quadrature distribution from the experiment. (b) Wigner function based on the output of the SDP.

that a lower bound to the nonclassicality of the phase randomized state will be valid for the original state.

In order to apply our algorithm we use the measured data to constrain all the points of the characteristic function on a 37×37 lattice. Error bars are estimated using Eq. (6) with $n = 5$ standard deviations. This means that the probability that all the points of the lattice lie inside the error bars is larger than 99.9%. The lower bound for the nonclassicality coming out from the SDP (200 random test vectors) is $x' \simeq 0.0028$, meaning that the Wigner function of the state cannot be a positive probability distribution. The Wigner function reconstructed from the optimal solution of the SDP [Fig. 2(b)] is clearly negative even if we asked for the most positive one consistent with measured data.

Extensions of this approach.—Needless to say, this approach can be extended in several ways. Indeed, the method can readily be generalized to produce lower bounds for *entanglement measures* [11] in the multimode setting. Also, this idea can be applied to the situation when not slices are measured, but *points in phase space*, such as when using a detector-atom that is simultaneously coupled to a cantilever [22]. It also constitutes an interesting perspective to apply the present ideas to certify deviations from *stabilizer states* for spin systems (as those states having a positive discrete Wigner function [28]).

Summary.—We have introduced a method to directly measure the nonclassicality of quantum mechanical modes, requiring less information than tomographic knowledge, but at the same time in a certified fashion. These ideas are further advocating the paradigm of "learning much from little"—getting much certified information from few measurements—complementing methods of *witnessing entanglement* [11,12], ideas of *compressed sensing* [29] or *matrix-product based* [30] approaches to quantum state tomography, *detector tomography* [31], or the direct estimation of *Markovianity* [32].

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[18]
$$\sigma = \begin{bmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{bmatrix}$$
reflects the commutation relations.

[19] See supplementary material at http://link.aps.org/supplemental/10.1103/PhysRevLett.106.010403.

[20] Since the error follows a Gaussian distribution, the probability P_{out} that the actual value of the characteristic function lies outside the error bar is never zero but it is exponentially suppressed for increasing n. E.g., for $n = 3$ one has $P_{\mathrm{out}} < 2.7 \times 10^{-3}$, for $n = 5$ one has $P_{\mathrm{out}} < 6 \times 10^{-7}$, etc..

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[26] Note that the approach followed here is general enough to accommodate further additional constraints, as long as they can be captured in terms of an SDP based on values of the characteristic function. In this way, one can also add specific figures of merit related to actual target states.

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