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Limitations of quantum computing with Gaussian cluster states

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We discuss the potential and limitations of Gaussian cluster states for measurement-based quantum computing. Using a framework of Gaussian-projected entangled pair states, we show that no matter what Gaussian local measurements are performed on systems distributed on a general graph, transport and processing of quantum information are not possible beyond a certain influence region, except for exponentially suppressed corrections. We also demonstrate that even under arbitrary non-Gaussian local measurements, slabs of Gaussian cluster states of a finite width cannot carry logical quantum information, even if sophisticated encodings of qubits in continuous-variable systems are allowed for. This is proven by suitably contracting tensor networks representing infinite-dimensional quantum systems. The result can be seen as sharpening the requirements for quantum error correction and fault tolerance for Gaussian cluster states and points toward the necessity of non-Gaussian resource states for measurement-based quantum computing. The results can equally be viewed as referring to Gaussian quantum repeater networks.

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I. INTRODUCTION

Optical systems offer a highly promising route to quantum information processing and quantum computing. The seminal work in Ref. [1] showed that even with linear optical gate arrays alone and appropriate photon counting measurements, efficient linear optical computing is possible. The resource overhead of this proof-of-principle architecture for quantum computing was reduced, indeed by orders of magnitude, by directly making use of the idea of measurement-based quantum computing with cluster states [2–4]. Such an approach is appealing for many reasons; the reduction of resource overhead is one, and the clear-cut distinction between the creation of entanglement as a resource and its consumption in computation is another. This idea was further developed into the continuous-variable (CV) version thereof [5–8], which aims at avoiding limitations related to efficiencies of creation and detection of single photons. In this context, Gaussian states play a quite distinguished role, as they can be created by passive optics, optical squeezers, and coherent states, i.e., the states produced by a usual laser [9–13]: Indeed, Gaussian cluster states are a promising resource for instances of quantum computing with light. Such a CV scheme allows for deterministic preparation of resource states, while schemes based on linear optics with single photons require preparation methods which are intrinsically probabilistic.

In this work, however, we highlight and flesh out some limitations of such an approach. We do so to clarify the exact requirements that any scheme for CV quantum computing based on Gaussian cluster states eventually will have to fulfill and what quantum error correction and fault-tolerant approaches eventually have to deliver. Specifically, we show that Gaussian local measurements alone will not suffice to transport quantum information across the lattice, even on complicated lattices described by an arbitrary graph of finite dimension: Any influence of local measurements is confined to a local region, except from exponentially suppressed corrections. This can be viewed as an impossibility of Gaussian error correction in the measurement-based setting. What is more, even under non-Gaussian measurements, this obstacle cannot be overcome, to transport or process quantum information along slabs of a finite width: Any influence of local measurements will again exponentially decay with the distance. This observation suggests that—although the initial state is perfectly known and pure—finite squeezing has to be tackled with a full machinery of quantum error correction and fault tolerance [14–16], yet developed for this type of system and, presumably, giving rise to a massive overhead. No local measurements or suitable sophisticated encodings of qubits in finite slabs—reminding, e.g. of encodings of the type of Ref. [16]—can uplift the initial state to an almost perfect universal resource. To arrive at this conclusion, in some ways, we explore ideas of measurement-based computing beyond the one-way model [2] as introduced in Ref. [17] and further developed in Refs. [18–22]. We highlight the technical results as "observations" and discuss implications of these results in the text. While these findings do not constitute a "no-go" argument for Gaussian cluster states, they do seem to require a very challenging prescription for quantum error correction and further highlight the need to identify alternative schemes for CV quantum computing, specifically schemes based on *non-Gaussian CV states*. *Small-scale implementations* of *Gaussian cluster-state computing*, as we will see, are also *not* affected by these limitations.

The structure of this article is as follows: In Sec. III we discuss the concept of Gaussian projected entangled pair states (GPEPSs), forming a family of states including the physical CV Gaussian cluster state. In Sec. IV we discuss the impact of Gaussian measurements on GPEPSs and show that under this restriction the localizable entanglement in every GPEPS decays exponentially with the distance between any two points on an arbitrary lattice. This also has implications for Gaussian quantum repeaters, which we investigate in detail. Then we leave the strictly Gaussian stage in Sec. V and present our main result, showing that under more general measurements

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of GPEPSs, quantum information processing in finite slabs is still not possible. We discuss requirements for error correction, before presenting concluding remarks.

II. PRELIMINARIES

A. Gaussian states

Before we turn to measurement-based quantum computing (MBQC) on CV states, we briefly review some basic elements of the theory of Gaussian states and operations which are needed in this article [9–12]. Readers familiar with these concepts can safely skip this section. Although the statements made in this work apply to all physical systems described by *quadratures* or *canonical coordinates*, including, for example, micromechanical oscillators, we have a *quantum optical system* in mind and often use language from this field as well. Any system of N bosonic degrees of freedom, for example, N light modes, can be described by canonical coordinates $x_n = (a_n + a_n^\dagger)/2^{1/2}$ and $p_n = -i(a_n - a_n^\dagger)/2^{1/2}$, $n = 1,\ldots,N$, where a_n ($a_n^\dagger$) annihilates (creates) a photon in the respective mode. When we collect these $2N$ canonical coordinates in a vector $O = (x_1, p_1, \ldots, x_N, p_N)$, we can write the commutation relations as $[O_j, O_k] = i\sigma_{j,k}$, where the *symplectic matrix* σ is given by

$$\sigma = \bigoplus_{j=1}^{N} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{1}$$

Gaussian states are fully characterized by their first and second moments alone. The *first moments* form a vector d with entries $d_j = \mathrm{tr}(O_j\rho)$, while the second moments, which capture the fluctuations, can be collected in a $2N \times 2N$ matrix γ, the so-called *covariance matrix* (CM), with entries

$$\gamma_{j,k} = 2\mathrm{Re}\ \mathrm{tr}\,[\rho(O_j - d_j)(O_k - d_k)]. \tag{2}$$

Hence, Gaussian states are complete characterized by d and γ. *Gaussian unitaries*, that is, unitary transformations acting in Hilbert space preserving the Gaussian character of the state correspond to symplectic transformations on the CM. They in turn correspond to maps $\gamma \mapsto S\gamma S^T$ with $S\sigma S^T = \sigma$. The set of such *symplectic transformations* forms the group $Sp(2N,\mathbb{R})$. A set of particularly important example Gaussian states are the *coherent states*, for which the state vectors read, in the photon number basis,

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \tag{3}$$

and are described by $d = \sqrt{2}(\mathrm{Re}\,\alpha, \mathrm{Im}\,\alpha)$ and $\gamma = \mathrm{diag}(1,1)$. Single-mode *squeezed states* are characterized by lower fluctuations in one phase-space coordinate. The CM can, in a suitable basis, then be written as $\gamma = \mathrm{diag}(x, 1/x)$ with $x \neq 0$.

B. MBQC on Gaussian cluster states

The first proposal for MBQC on CV states has been based on so-called Gaussian cluster states and works in almost-complete analogy to the qubit case [5–8]. As such, the formulation is based on "infinitely squeezed" and hence unphysical states using infinite energy in preparation: It can be created by initializing every mode in the $p = 0$ "eigenstate" of p (formally an improper eigenstate of momentum, a concept that can be made rigorous, for example, in an algebraic formulation [23]). This is the CV analog to the state vector $|+\rangle = (|0\rangle + |1\rangle)/2^{1/2}$ in the qubit case. Then the operation $e^{ix\otimes x}$, the analog to the C_Z gate, is applied between all adjacent modes. This state allows universal MBQC to be performed with Gaussian and one non-Gaussian measurement. The state as such is not physical and not contained in Hilbert space. The argument, however, is that it should be expected that a finitely squeezed version inherits essentially the same properties. Replacing them by finitely squeezed ones, we obtain a state which we call a *physical Gaussian cluster state*.

III. GPEPS

Projected entangled pair states or *tensor product states* have been used for qubits to generalize matrix product states or *finitely correlated states* [24,25] from one-dimensional (1D) chains to arbitrary graphs [26–28]. One suitable way to define them is via a valence-bond construction: One can create a state by placing entangled pairs—constituting "virtual systems"—on every bond of the lattice and then applying a suitable projection to a single mode at every lattice site. These projections, often taken to be equal, together with the specification of the initial entangled states, then serve as a description of the resulting state. *Matrix product states for Gaussian states* (MPSGs) have been studied to obtain correlation functions and entanglement scaling in 1D chains [29].

In this work we focus on GPEPSs which can be obtained from non–perfectly entangled pairs. The bonds we consider are *two-mode squeezed states* (TMSSs), the state vectors of which have the photon number representation

$$|\psi_\lambda\rangle = (1-\lambda^2)^{1/2} \sum_{n=0}^{\infty} \lambda^n |n,n\rangle, \tag{4}$$

where $\lambda \in (0,1)$ is the *squeezing parameter*. We denote the corresponding density matrix ρ_λ. For $\lambda \to 1$ the state becomes "maximally entangled," but this limit is not physical because it is not normalizable and has infinite energy as already mentioned. We, therefore, carefully analyze the effects stemming from the fact that $\lambda < 1$. The CM of this state reads

$$\gamma_\lambda = \begin{bmatrix} \cosh(2r) & 0 & \sinh(2r) & 0 \\ 0 & \cosh(2r) & 0 & -\sinh(2r) \\ \sinh(2r) & 0 & \cosh(2r) & 0 \\ 0 & -\sinh(2r) & 0 & \cosh(2r) \end{bmatrix}, \tag{5}$$

where $\tanh(r/2) = \lambda$. This number r is also referred to as the *squeezing parameter* when there is no risk of mistaking one for the other. It is also known that any pure bipartite multimode Gaussian state can be brought into the tensor product of a TMSS [10,30] by means of local unitary Gaussian operations, each having a CM in the form of Eq. (5). Then the largest r in the vector of the resulting TMSS is referred to as its squeezing parameter.

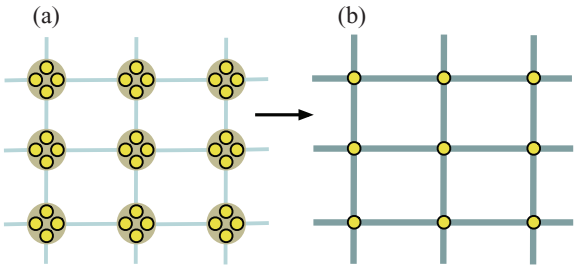

FIG. 1. (Color online) GPEPS on an arbitrary graph, here one representing a cubic lattice. (a) Connected dots represent two-mode squeezed states; circles denote vertices where Gaussian projections are being performed. (b) The resulting GPEPS after local Gaussian projections have been performed on the virtual systems. Any Gaussian cluster state can be prepared in this fashion.

We also discuss GPEPSs on general graphs $G = (V, E)$, as shown in Fig. 1. Vertices G here correspond to physical systems, and edges E to connections of neighborhood. In any such graph, $d(.,.)$ is the natural graph-theoretical distance between two vertices. As we often consider the system of bonds before the projection operation is performed, we employ the following notation: We speak of operations on *virtual systems* when referring to collective operations on modes before the projection is applied and often emphasize this when speaking of a single physical system with Hilbert space $\mathcal{H} = \mathcal{L}^2(\mathbb{R})$. Note that we also allow for more than one edge between two vertices in a graph.

When a particular vertex has N adjacent bonds, the projection map is a Gaussian operation of the form

$$V : \mathcal{H}^{\otimes N} \to \mathcal{H}. \tag{6}$$

This operation can always be made trace preserving [9,12,31,32], in quite sharp contrast to the situation in the finite-dimensional setting. This operation is also referred to as GPEPS projection. This operation can always be realized by mixing single-mode squeezed states on a suitably tuned beam splitter, which means that *inline* squeezers are not necessary [33]. Note that any such state can also be used as a variational state to describe ground states of many-body systems and, by construction, satisfies an *entanglement area law* [34].

IV. GAUSSIAN OPERATIONS ON A GPEPS

In this section, we discuss Gaussian operations on a GPEPS and derive some statements on entanglement swapping, the localizable entanglement, and the usefulness as a resource for MBQC. Since all measurements are assumed to be Gaussian as well, this is, as such, not yet a full statement on universality, but already shows that the natural operation for transport of logical information in such a Gaussian cluster state does not work with such local measurements.

A. Localizable entanglement

The *localizable entanglement* between two sites A and B in the graph $G = (V, E)$ is defined by the maximal entanglement obtainable on average when performing projective measurements at all sites but A and B [35]. When we require both the initial state and the measurements to be Gaussian [36,37], the situation is simplified, as the entanglement properties do not depend on the measurement outcomes [9,12,31,32]. Thus, we do not need to average, but only to find the best measurement strategy. To be specific, we measure the entanglement in terms of the *logarithmic negativity*, which can be defined as [38–40]

$$E(\rho) = \log_2 \|\rho^{T_A}\|_1, \tag{7}$$

where T_A denotes the *partial transpose* with respect to subsystem A and $\|\cdot\|_1$ the trace-norm, and we use the natural logarithm. For a TMSS, E coincides with the squeezing parameter as $E(\rho_\lambda) = r$. It is important to note, however, that this choice has only been made for notational convenience: In our statements on asymptotic degradation of entanglement, any other measure of entanglement would also do, specifically the entropy of entanglement for pure Gaussian states and the *distillable entanglement* or the *entanglement cost* for mixed states.

We mostly focus on two variants of the concept of localizable entanglement: Whenever we allow only for Gaussian local measurements, we refer to this quantity as *Gaussian localizable entanglement*, abbreviated E_G. Then we consider the situation where we ask for fixed subspaces S_A and S_B in the Hilbert spaces associated with sites A and B to become entangled by means of local measurements. We then refer to *subspace localizable entanglement* E_S. Both concepts directly relate to transport in MBQC.

B. Entanglement swapping

The task of localizing entanglement in a PEPS is closely related to that of *entanglement swapping* [41]. In this situation we have three parties, A, B, and C, where both A and B and B and C share an entangled pair each. Then B, consisting of B_1 and B_2, is allowed to perform an arbitrary Gaussian operation on its parts of the two pairs, followed by a measurement. The task is to choose the operation in such a way that the resulting entanglement between A and B is maximum.

Lemma 1. Optimality of Gaussian Bell measurement for entanglement swapping of TMSSs. For two pairs of entangled TMSSs shared between A and B_1 and between B_1 and C, the supremum of maximum achievable negativity between A and C by a local Gaussian measurement in B_1, B_2 is approximated by the measurement that best approximates a Gaussian Bell measurement.

We consider the situation of having a TMSS (5)

$$|\psi\rangle_{A,B_1} = \left|\psi_{\lambda_1}\right\rangle_{A,B_1}, \quad |\psi\rangle_{B_2,C} = \left|\psi_{\lambda_2}\right\rangle_{B_2,C} \tag{8}$$

with some $\lambda_1, \lambda_2 > 0$ and restricting the operation on B to be Gaussian. Furthermore, we allow for operations which do not succeed with unit probability. We have to allow for general local Gaussian operations and, also, for arbitrary local additional Gaussian resources, with CM γ_B on mode B_3, on an arbitrary number of modes. The initial CM of the system hence reads

$$\gamma = \gamma_{\lambda_1} \oplus \gamma_{\lambda_2} \oplus \gamma_{B_3}. \tag{9}$$

Without loss of generality, one can assume that one performs a single projection onto a pure Gaussian state on all modes

referring to B. Ordering modes to A,C,B_1,B_2,B_3, one can write the CM in block form as

$$\gamma = \begin{bmatrix} U & V & 0 \\ V^T & W & 0 \\ 0 & 0 & \gamma_{B_3} \end{bmatrix}, \tag{10}$$

with U referring to A, C and V referring to B_1,B_2. When we project the modes B_1, B_2, and B_3 onto a pure Gaussian state with CM Γ, the CM of the resulting state of A and C, postselected for that outcome, is given by the Schur complement [9,31,32],

$$\gamma_{A,C} = \begin{bmatrix} U & 0 \\ 0 & 0 \end{bmatrix} - [V \quad 0] \left(\begin{bmatrix} W & 0 \\ 0 & \gamma_{B_3} \end{bmatrix} + \Gamma \right)^{-1} \begin{bmatrix} V^T \\ 0 \end{bmatrix}. \tag{11}$$

Any symplectic operation S applied to B before the measurement can, of course, also just be absorbed into the choice of the CM Γ. Writing

$$\begin{bmatrix} W & 0 \\ 0 & \gamma_{B_3} \end{bmatrix} + \Gamma = \begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix}, \tag{12}$$

one finds that the upper-left principal submatrix of the inverse can be written as

$$\begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix}^{-1} \Big|_{B_1,B_2} = (X - YZ^{-1}Y^T)^{-1}, \tag{13}$$

again, in terms of a Schur complement expression. Since $\gamma_{B_3} + i\sigma \geqslant 0$ and the same holds for the subblock on B_3 of Γ, these matrices are clearly positive. Using operator monotonicity of the inverse function, one finds that

$$(X - YZ^{-1}Y^T)^{-1} \geqslant 0 \tag{14}$$

holds, since $YZ^{-1}Y^T \geqslant 0$. Therefore,

$$\gamma_{A,C} = \gamma'_{A,C} + P, \tag{15}$$

with a matrix $P \geqslant 0$. Here $\gamma'_{A,C}$ is the CM following the same protocol, but where Γ is replaced by an identical CM, but with $Y = 0$. To arrive at such a CM is always possible and still gives rise to a valid CM by virtue of the pinching inequality. This is still merely the CM of the Gaussian state, subjected to additional classically correlated Gaussian noise. In other words, it is always optimal to treat B_3 as an innocent bystander and not to perform an entangling measurement between B_1 and B_2, on the one hand, and B_3, on the other hand: quite consistent with what one could have intuitively assumed. We can hence focus on the situation where B_3 is absent and we merely project onto a pure Gaussian state in B_1 and B_2.

It is then easy to see that there is no optimal choice, but the supremum can be better and better approximated by considering more and more squeezed TMSSs (or "infinitely squeezed states" in the first place), that is, on $|\psi_\lambda\rangle$ in the limit of $\lambda \to 1$, which is the CV analog to the Bell state for qudits. This measurement can be realized by mixing B_1 and B_2 on a beam splitter with reflectivity $R = 1/2$ and performing homodyne measurement on both modes afterward (i.e., a projection on an infinitely squeezed single-mode state being an improper eigenstate of the position operator). From Eqs. (5) and (11) with $\Gamma = \gamma_\lambda$ and performing the limit $\lambda \to 1$, we can calculate the CM of the resulting state. It has the form of (5), with

$$r = f(r_1,r_2) = \frac{1}{2}\operatorname{arcosh}\frac{1 + \cosh 2r_1 \cosh 2r_2}{\cosh 2r_1 + \cosh 2r_2}. \tag{16}$$

We note that f is symmetric in its arguments and fulfills $f(r_1,r_2) < \min\{r_1,r_2\}$ and $\lim_{r_1\to\infty} f(r_1,r_2) = r_2$. This means that arbitrarily faithful entanglement swapping is possible exactly in the limit of infinite entanglement. Otherwise, the entanglement necessarily deteriorates [41].

To show that this measurement is indeed optimal, we set

$$\Gamma = S\gamma_\lambda S^T, \tag{17}$$

where $S \in Sp(4,\mathbb{R})$. Calculating the resulting degree of entanglement, a direct and straightforward inspection reveals that $E(\rho_{A,C})$ can only decrease whenever we choose $S \neq \mathbb{1}$.

C. 1D chain

We now turn to a one-dimensional GPEPS, not allowing multiple bonds in the valence-bond construction, and are in the position to show the following observation.

Observation 1. Exponential decay of Gaussian localizable entanglement in a 1D chain. Let G be a 1D GPEPS, and A and B two sites. Then

$$E_G(A,B) \leqslant c_1 e^{-d(A,B)/\xi_1}, \tag{18}$$

where $c_1,\xi_1 > 0$ are constants. The best performance is reachable by passive optics and homodyning only.

To prove this, we interpret the preparation projection (6) and the following measurements of the localizable entanglement protocol as a sequence of instances of entanglement swapping. Clearly, to allow for general Gaussian projections is more general than using (i) the specific Gaussian projection of the PEPS, followed by a (ii) suitable Gaussian projection onto a single mode; hence every bound shown for this setting will also give rise to a bound to the actual 1D Gaussian chain. If $d(A,B)$ is again the graph-theoretical distance between A and B, we have to swap $k = d(A,B) - 1$ times. Defining $g(r) = f(r,r_I)$, where r_I is the initial strength of all bonds, and iterating the argument, we obtain

$$r_{A,B} = (g^{\circ k})(r_I) = F(k). \tag{19}$$

As the negativity is up to a simple rescaling equal to this two-mode squeezing parameter, the only task left is to show that $F(k)$ decays exponentially. To do this, we need $\operatorname{arcosh}(x) = \log_2[x + (x^2-1)^{1/2}]$ and the following relations which hold for $x \geqslant 0$: $\cosh(x) \geqslant e^x/2$ and $\cosh(x) \leqslant e^x$. With the help of these, we can conclude that

$$F(k+1)/F(k) < Q < 1 \tag{20}$$

for a Q depending only on r_I. Thus, $F(k)$ decays exponentially, which proves Observation 1. Note that to maximize the entanglement between A and B, we have chosen the supremum of the maps better and better approximating the projection onto an infinitely entangled TMSS. Thus, for a specific GPEPS which is characterized by a fixed map V, the E_G is generally lower.

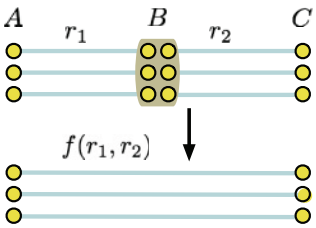

FIG. 2. (Color online) The situation referred to in Lemma 2. The strongest bonds before the projection are r_1 and r_2. The most significantly entangled bond has the strength $f(r_1,r_2)$.

This result has a remarkable consequence for Gaussian *quantum repeater lines*: It is not possible to build a 1D quantum repeater relying on Gaussian states, if only local measurements and no distillation steps are being used. We show in Sec. V that even non-Gaussian measurements cannot improve the performance. If one sticks to the Gaussian setting, also relying on complex networks does not remedy the exponential decay, as we see. Of course, non-Gaussian distillation schemes can be used to realize CV quantum repeater networks.

D. General graphs in arbitrary dimensions

One should suspect that the exponential decay of E_G is a special feature of the 1D situation and that higher dimensional graphs would eventually allow localization of a constant amount of entanglement. In this section we show that this is not the case. We first need a lemma which follows directly from our discussion of entanglement swapping.

Lemma 2. Collective operations on pure Gaussian states. Let ρ_{A,B_1} be a pure Gaussian state on $\mathcal{H}^{\otimes 2n}$ of n modes, and $\rho_{B_2,C}$ a pure Gaussian $\mathcal{H}^{\otimes 2m}$ state, where one part of each is held by A, B, and C, respectively (see Fig. 2). Let the maximum two-mode squeezing parameter be r_1 between A and B and r_2 between B and C. Then the maximum two-mode squeezing parameter achievable with a Gaussian projection in B between A and C is $f(r_1,r_2)$.

To prove this, we again use the fact that any two-party multimode pure Gaussian state can be transformed by local unitary Gaussian operations on both parties into a product of the TMSS [10,30]. This is nothing but the Gaussian version of the *Schmidt decomposition*. It hence does not restrict generality to start from this situation. As already noted, the best strategy for entanglement swapping between two pairs is a Gaussian Bell measurement, where the squeezing parameter changes according to f.

We now allow for global Gaussian operations on all subsystems belonging to B. We relax this situation to the following, where we allow for even more general operations: namely, a local Gaussian operation onto all modes of B, as well as onto all modes of A and C that are not the two modes that share the largest r. Clearly, this is a more general map than is actually considered in the physical situation. This, however, is exactly the situation already considered: an entanglement swapping scheme with an unentangled bystander. Hence, we again find that to project each pair onto a two-mode pure Gaussian state is optimal. For that, the sequence of projections better and better approximating an infinitely squeezed TMSS gives rise to the supremum. Hence, we have shown the preceding result. Now we can prove a central result of this work.

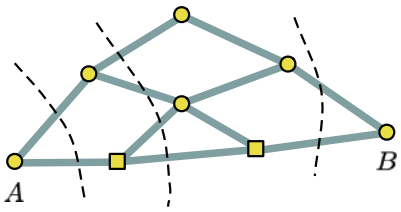

FIG. 3. (Color online) Partitioning of the graph according to the shortest path as described in the text. Sites drawn as squares are the those which lie on the shortest path connecting A and B.

Observation 2. Exponential decay of Gaussian localizable entanglement of a GPEPS in a general graph. Consider a GPEPS in a general graph with finite dimension and let A and B be two vertices of this graph. Then there exist constants $c_2,\xi_2 > 0$ such that

$$E_G(A,B) \leqslant c_2 e^{-d(A,B)/\xi_2}. \tag{21}$$

We take the shortest path between A and B—achieving the graph-theoretical distance $d(A,B)$—and denote its vertices $A, v_1, \ldots, v_{d(A,B)-1}, B$. We partition the graph in such a way that the boundaries do not intersect or touch each other and every vertex on the shortest path from A and B is contained in one region, which is called R_v (see Fig. 3). Again, we consider the situation of having TMSSs distributed in the graph between vertices sharing an edge—a general local Gaussian measurement on a GPEPS—so the GPEPS projection, now on several modes, followed by a specific single-mode Gaussian measurement, can only be less general than a general collective Gaussian measurement; thus, we again arrive at a bound to the localizable entanglement in the GPEPS.

Now we face exactly the situation to which Lemma 2 applies. In fact, in each step in each of the parts A, B, and C, we will have a collection of TMSSs, shared across the cut of the three regions. If r_{Av_1} is the strongest bond, in terms of the two-mode squeezing parameter, between R_A and R_{v_1}, and $r_{v_1v_2}$ is the strongest bond between R_{v_1} and R_{v_2}, then the strongest bond between R_A and R_{v_2} is given according to Lemma 2 by $f(r_{Av_1},r_{Av_2})$. Now we can proceed exactly as in the proof of Theorem 1—and again, any uncorrelated bystanders will not help to improve the degree of entanglement—and thus show Theorem 2. This again has a consequence for quantum repeaters: Even when an arbitrary number of parties can share arbitrary many Gaussian entangled bonds, it is not possible to teleport quantum information over an arbitrary distance, as shown here.

In fact, using this statement, one can show that any impact of measurements in terms of a measurable signal is confined to a finite region in the graph, with I now being a subset of the graph, except from exponentially suppressed corrections. This region could be a poly-sized region in which the input to the computation is encoded. The readout of the quantum computation is then estimated from measurements in some region O, giving rise to a bit that is the result of the original decision problem to be solved by the quantum computation. From the decay of localizable entanglement, it is not difficult to show that the probability distribution

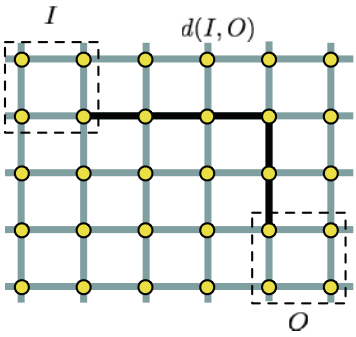

FIG. 4. (Color online) Exponential decay of any influence of any measurements in region I on statistics of measurement outcomes in region O in the graph-theoretical distance $d(I,O)$ between the regions.

of this bit is unchanged by measurements in I, except from corrections that are exponentially decaying with $d(I,O)$ (see Fig. 4).

Note that concerning small-scale "proof-of-principle" applications, the arguments presented do *not* impose a fundamental restriction, as they apply only to the situation where entanglement distribution over an arbitrary number of modes (or repeater stations) is required. For any *finite* distance $d(A,B)$ and required entanglement $E(A,B)$, there exists a *finite* minimal squeezing $\lambda_{\min}$ which allows performance of the task. Only asymptotically will one necessarily encounter this situation. The result can equally be viewed as the impossibility of Gaussian quantum error correction in a measurement-based setting, complementing the results in Ref. [42].

E. Remarks on Gaussian repeater networks

These results of course also apply to general *quantum repeater networks*, where the aim is to end up with a highly entangled pair between any two points in the repeater network (see, e.g., Ref. [43] for a qubit version thereof). That is, in Gaussian repeater networks, one will also need non-Gaussian operations to make the network work, quite consistent with the findings in Refs. [9,31,32].

F. MBQC

The impossibility of encountering a localizable entanglement that is not exponentially decaying directly leads to a statement on the impossibility of using a GPEPS as a quantum wire. Such a wire should be able to perform the following task [17]: Assume that a single mode holds an unknown qubit in an arbitrary encoding; that is,

$$|\phi_{\text{in}}\rangle = \alpha|0_L\rangle + \beta|1_L\rangle. \tag{22}$$

This system is then coupled to a defined site A, the first site of the wire, of a GPEPS by a fixed in-coupling unitary operation which can in general be non-Gaussian. To complete the in-coupling operation, the input mode is measured in an arbitrary basis, where we also allow for probabilistic protocols; that is, the operation does not have to succeed for all measurement outcomes. Then one performs local Gaussian measurements on each of the modes. Then, at the end, one expects the mode at a single site B to be in the state vector $|\phi_{\text{out}}\rangle = U|\phi_{\text{in}}\rangle$ (or at least arbitrarily close in trace norm) for any chosen $U \in \text{SU}(2)$. Note that the length of the computation, and therefore the position of output mode B, may vary and that the computational subspace can be left during the measurement. We want to stress that it is also possible to consider quantum wires which process qudits or even CV quantum information, where even on the logical level, information is encoded continuously. However, the capability of processing a qubit is clearly the weakest requirement. Thus, we address this situation only because the corresponding statements for other quantum wires immediately follow. With this clarification we can state the following lemma.

Observation 3. Impossibility of using Gaussian operations on arbitrary GPEPSs in general graphs for quantum wires. No GPEPS on any graph together with Gaussian measurements can serve as a perfect quantum wire for even a single qubit.

This is obvious from the previous considerations, as the measurements for the localizable entanglement and the incoupling operation commute, and clearly, the procedure is especially not possible for $U = \mathbb{1}$. The same argument, of course, also holds true in general graphs: No wire can be constructed from local Gaussian measurements in this sense, again for an exponential decay of the localizable entanglement. This observation is related to the decay of fidelity when performing CV quantum teleportation with squeezed vacuum states, as discussed in Ref. [44]. As mentioned, this statement can also be refined to having up to exponential corrections of finite-influence regions altogether.

V. NON-GAUSSIAN OPERATIONS

We now turn to our second main result, namely, that—under rather general assumptions which we detail below—Gaussian states defined on slabs of a finite width cannot be used as perfect primitives for resources for MBQC, even if non-Gaussian measurements are allowed for: Any influence of local measurements will again exponentially decay with distance.

More specifically, we first show that a 1D GPEPS cannot constitute a quantum wire in the sense of the definition in Sec. IV F extended to arbitrary measurements. This already covers all kinds of sophisticated encodings that can be carried by a single quantum wire, including ideas of "encoding qubits in oscillators" [16]. We then discuss the situation where an entire cubic slab of constant width is being used to encode a single quantum logical degree of freedom and find that the fidelity of transport will still decay exponentially. Not even using many modes and coupled quantum wires, possibly employing ideas of distillation, can this obstacle be overcome with local measurements alone. That is, we show that Gaussian states cannot be uplifted to serve as perfect universal resource states by measurements on finite slabs alone: Frankly, the finite squeezing present in the initial resources—although the state is pure and known—must be treated as a faulty state, and some full machinery of *fault tolerance* [14,15], which has yet to be developed for this kind of system, necessarily has to be applied even in the absence of errors. This contrasts quite severely with other limitations known for Gaussian quantum states. For example, while the distillation of entanglement is

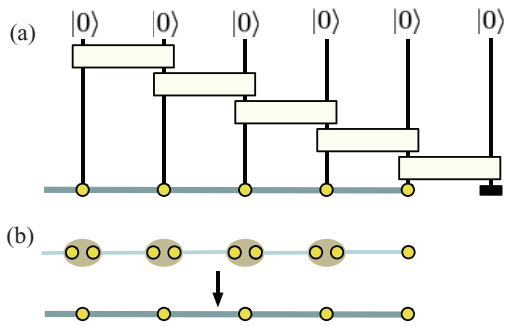

FIG. 5. (Color online) (a) Sequential preparation of a GMPS state: Each line represents a mode of a unitary tensor network, whereas each box stands for a Gaussian unitary. For a suitable choice of Gaussian unitaries, the resulting state is a Gaussian cluster state being prepared in the valence-bond construction (b).

not possible using Gaussian operations alone, non-Gaussian operations help to accomplish this task [45].

A. Sequential preparation of 1D Gaussian quantum wires

To make the statement, we first have to introduce another equivalent way of defining a GPEPS— or, specifically, a GMPS—in one dimension: It is easy to see that a GMPS with state vector $|\psi\rangle$ of N modes can be prepared as

$$|\psi\rangle = \langle\omega|_{N+1} \prod_{j=1}^{N} U^{(j,j+1)} |0\rangle^{\otimes(N+1)}, \tag{23}$$

with identical Gaussian unitaries $U^{(j,j+1)}$ supported on modes $j, j+1$, depicted as gray bars in Fig. 5. This follows immediately from the original construction in Ref. [24] (see also Ref. [25]), translated into the Gaussian setting. A detailed study of sequentially preparable infinite-dimensional quantum systems with an infinite or finite bond dimension will be presented elsewhere.

B. Impossibility of transport by non-Gaussian measurements in one dimension: General considerations

We start by stating the main observation here: Frankly, even under general non-Gaussian measurements, transport along a 1D chain is not possible. We refer to the notions of both localizable entanglement and the *probability of transport*: This is the average maximum probability of recovering an unknown input state in a fixed subspace S of dimension at least $\dim(S) \geqslant 2$ which has been transported through the wire: Specifically, one asks for the maximum average success probability of a positive operator-valued measure applied to the output of the wire that leads to the identity channel up to a constant, where the average is taken with respect to all possible outcomes when performing local measurements transporting along the wire. We see that this probability decays exponentially with the distance between the input and the output site.

This decay follows regardless of the encoding chosen. Note that by no means do we require logical information to be contained in a certain fixed logical subspace along the computation: Only in the first and last steps—when initially encoding quantum information or coupling to another logical qubit—do we ask for a fixed subspace. This logical subspace is even allowed to stochastically fluctuate along the computation dependent on measurement outcomes that are obtained in earlier steps of the computation.

Observation 4. Impossibility of using Gaussian 1D chains as quantum wires under general measurements. Let G be a one-dimensional GPEPS. Let S be either $S = \mathcal{H}$ or a subspace thereof. Then the probability of transport between any two sites A and B of the wire satisfies

$$p \leqslant c_3 e^{-d(A,B)/\xi_3} \tag{24}$$

for suitable constants $c_3, \xi_3 > 0$. This implies that for any subsets of sites E_A and E_B and for fixed local subspaces, the entanglement between E_A and E_B that can be achieved by arbitrary local measurements of all sites except those contained in E_A and E_B is necessarily exponentially decaying in $d(E_A, E_B)$. This also means that, for any two sites A and B,

$$E_S(A,B) \leqslant c_4 e^{-d(A,B)/\xi_3} \tag{25}$$

for some $c_4 > 0$ are constants, even if arbitrary local measurements are taken into account.

We now proceed in two steps. First, it is shown that there exists no subspace $S \in \mathcal{H}$ of dimension at least $\dim(S) \geqslant 2$ such that V_j can be chosen to be unitary, for all j for which $p_j > 0$ and

$$\langle\eta_j|U|\psi\rangle|0\rangle = p_j^{1/2} V_j|\psi\rangle \tag{26}$$

for all $|\psi\rangle \in S$, where all U is the Gaussian unitary of the sequential preparation in Eq. (23), where the index of the mode, and also any label of tensor factors, is suppressed (see Fig. 6). $\{|\eta_j\rangle\}$ is an orthonormal basis of $\mathcal{H}$, with j labeling the respective outcome of the local measurement, possibly a continuous function. Because the computational subspace S is allowed to vary during the processing but must be invariant for the computation as a whole, we have to consider all N steps in the sequential preparation and all measurements together. For reasons of simplicity, we present the argument for a wire consisting of just two sites first and extend it afterward. We

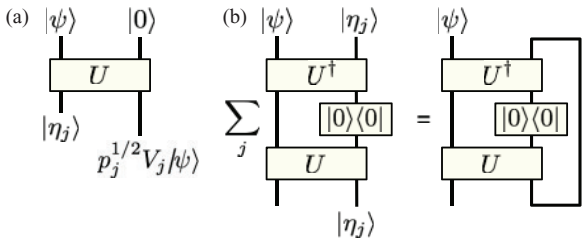

FIG. 6. (Color online) (a) Network representing a single step of a sequential preparation of a GMPS; (b) tensor network representation of $\langle\psi|\langle 0|U^\dagger(\mathbb{1}\otimes|0\rangle\langle 0|)U|0\rangle|\psi\rangle$.

define the operator

$$M = U^\dagger(\mathbb{1} \otimes |0\rangle\langle 0|)U \tag{27}$$

and formulate the subsequent lemma.

Lemma 3. Conditions for nondecaying transport fidelity. A necessary condition for Eq. (26) to be satisfied is that

$$\langle\psi|\langle\eta_j|M|\psi\rangle|\eta_j\rangle = p_j \tag{28}$$

for all j and all $|\psi\rangle \in S$, with $\sum_j p_j = 1$ and $\{|\eta_j\rangle\}$ forming a complete orthonormal basis of $\mathcal{H}$.

To see this, note that the fact that Eq. (26) holds true for each j for any $|\psi\rangle \in S$ means that

$$P_S\langle\eta_j|U|0\rangle P_S = p_j^{1/2} P_S, \tag{29}$$

where P_S denotes the projection onto S. Using completeness of $\{|\eta_j\rangle\}$,

$$\sum_j |\eta_j\rangle\langle\eta_j| = \mathbb{1}. \tag{30}$$

A moment of thought reveals that for any $|\phi\rangle \in S^\perp$, the latter denoting the orthogonal complement of S, one has that

$$P_S\langle\eta_j|U|\phi\rangle|0\rangle = 0. \tag{31}$$

What is more,

$$\langle\phi|\langle\eta_j|U|0\rangle P_S = 0, \tag{32}$$

again, for all $|\phi\rangle \in S^\perp$. This further means that (see Fig. 6)

$$\langle\psi|\langle\eta_j|U^\dagger(\mathbb{1} \otimes |0\rangle\langle 0|)U|\psi\rangle|\eta_j\rangle = \langle\psi|\langle\eta_j|M|\psi\rangle|\eta_j\rangle = p_j, \tag{33}$$

which proves Lemma 3. Now summing over all measurement outcomes j in Eq. (33), which is the same as performing the partial trace (see Fig. 6) with respect to the second mode, we obtain

$$\langle\psi|\mathrm{tr}_2\big(U^\dagger(\mathbb{1} \otimes |0\rangle\langle 0|)U\big)|\psi\rangle = 1, \tag{34}$$

which in turn implies, together with the preceding, that

$$P_S \mathrm{tr}_2[U^\dagger(\mathbb{1} \otimes |0\rangle\langle 0|)U]P_S = P_S. \tag{35}$$

But this in turn means that the Gaussian operator $\mathrm{tr}_2[U^\dagger(\mathbb{1} \otimes |0\rangle\langle 0|)U]$ has at least two spectral values that are identical. Now it is only possible for a Gaussian operator to have two equal, nonzero spectral values if the spectrum is flat and corresponds to an operator that is not of trace class (related to "infinite squeezing" and "infinite energy," which was excluded due to the restriction to proper quantum states with finite energy).

We now extend the argument to a wire of arbitrary length. Toward this aim we denote the measurement basis on the kth site $\{|\eta_j^{(k)}\rangle\}$ and the corresponding probabilities $p_j^{(k)}$. Definition (27) is generalized to

$$M = [U^\dagger(\mathbb{1} \otimes |0\rangle)]^N[(\langle 0| \otimes \mathbb{1})U]^N. \tag{36}$$

Condition (26) becomes

$$\big(\otimes_k \langle\eta_j^{(k)}|\big)U^{\otimes N}|\psi\rangle|0\rangle^{\otimes N} = \prod_k \big(p_j^{(k)}\big)^{1/2} V_j^{(k)}|\psi\rangle, \tag{37}$$

where $\prod_k V_j^{(k)}$ is unitary for all sequences of measurement outcomes and, furthermore, acts trivially on $S^\perp$. Modifying also Eqs. (32), (33), and (35) in a similar manner and using the completeness of the N measurement bases $\{|\eta_j^{(k)}\rangle\}$, we find that for Eq. (37) to hold, the Gaussian operator $O = \mathrm{tr}^N(M)$, where tr^N denotes the N-fold partial trace (or suitable tensor contraction), has two equal spectral values, which is not possible, as already mentioned, and thus, the first step in the proof is complete.

C. Impossibility of transport by non-Gaussian measurements in one dimension: Proving a gap

In the second step we now show that Observation 4 holds if Eq. (26) is not fulfilled. The problem of recovering an unknown state after propagation through the wire is equivalent to that of undoing a nonunitary operation. Obviously, it is a fundamental feature of quantum mechanics that it is not possible to implement a nonunitary linear transformation in a deterministic fashion. Since one does not have to correct for a nonunitary operation in each step, however, the technicality of the argument is related to the fact that we only have to undo an entire sequence of nonunitary Kraus operators once.

Assume that we aim to use our wire for the transport of a single pure qubit. After N steps of transport it will still be pure but, in general, distorted, due to the application of some nonunitary operator

$$V_J = V_{j_N}^{(N)} \cdots V_{j_1}^{(1)}, \tag{38}$$

where $J = (j_1, \ldots, j_N)$ is an index reflecting the entire sequence of measurement outcomes on the N lattice sites. To recover the initial state, one has to apply an X_J such that

$$X_J V_J = c_J \mathbb{1}, \tag{39}$$

with $c_j \in \mathbb{C}$. The success probability of this recovery operation, averaged over all measurement outcomes, is nothing but the *probability of transport*. It will decay exponentially in N whenever, for any k, at least a single $V_{j_k}^{(k)}$ is not unitary. The maximal average probability to undo random sequence V_J of Kraus operators is found to be

$$p_N = \max \mathrm{tr}(X_J V_J \rho V_J^\dagger X_J^\dagger), \tag{40}$$

subject to

$$X_J^\dagger X_J = \mathbb{1}, \tag{41}$$

$$X_J V_J = c_J \mathbb{1}. \tag{42}$$

A moment of thought reveals that this probability of transport is then found to be

$$p_N = \sum_J \lambda_1[(V_J^\dagger V_J)^{-1}]^{-1} = \sum_J \lambda_n(V_J^\dagger V_J), \tag{43}$$

where λ_1 (λ_n) denotes the largest (smallest) eigenvalue.

To show that Observation 4 is true if V_k is not proportional to a unitary matrix for at least one k can be shown by induction. Denoting, again, the operators applied by the measurements of the first N sites by V_J and the corresponding operators for

site $N+1$ by $\{W_j\}$, we get from Eq. (43) that

$$p_{N+1} = \sum_{J,j} \lambda_n (V_J^\dagger W_j^\dagger W_j V_J). \tag{44}$$

Before we proceed, we note that it is possible to assume that all W_j and V_j are effective 2×2 matrices, corresponding to the situation where the computational subspace S does not change. If this is not the case, one can account for the fluctuation of the computational subspace by replacing $V_j \mapsto U_j V_j$ (and performing an analogous replacement for W_j) with a suitable unitary U_j. All arguments that follow do not depend on the choice of this unitary U_j. Key to the exponential decay is a lemma that is proven in the Appendix.

Lemma 4. Bound to eigenvalues of the sum of 2×2 *matrices.* For any positive $A, B \in \mathbb{C}^{2\times 2}$ with $[A,B] \neq 0$, there exists a $\delta > 0$ such that

$$\lambda_2(A+B) \geqslant \lambda_2(A) + \lambda_2(B) + \delta. \tag{45}$$

If there exists at least one pair (i,j) for which

$$[W_i^\dagger W_i, W_j^\dagger W_j] \neq 0, \tag{46}$$

then also

$$[V_J^\dagger W_i^\dagger W_i V_J, V_J^\dagger W_j^\dagger W_j V_J] \neq 0, \tag{47}$$

and we can apply Lemma 4 directly to Eq. (44). If, in contrast,

$$[W_i^\dagger W_i, W_j^\dagger W_j] = 0 \tag{48}$$

for all pairs (i,j), all $W_i^\dagger W_i$ can be simultaneously diagonalized. This means that we can—without loss of generality—assume that

$$W_i^\dagger W_i = \mathrm{diag}(\xi_i, \zeta_i). \tag{49}$$

Because a nonunitary W_i exists by assumption, $\min\{|\xi_i - \zeta_i| : i = 1,2\} > 0$. In both cases we are provided with a $\nu < 1$ such that

$$p_{N+1} \leqslant \nu \sum_J \lambda_2(V_J^\dagger V_J) = \nu p_N, \tag{50}$$

where we have used the completeness relation

$$\sum_j W_j^\dagger W_j = \mathbb{1}. \tag{51}$$

This observation gives rise to the anticipated gap that proves the exponential decay of the probability of transport and, therefore, to Observation 4. The exponential decay of the subspace localizable entanglement follows directly: If there was a nondecaying localizable entanglement, this could be used to transport with a high recovery probability, in contrast to what we have shown. If this were not the case, one could use the wire to distribute entanglement, which is obviously not possible.

D. Impossibility of transport by non-Gaussian measurements in one dimension: Concluding remarks

Note, finally, that even though we have presented Observation 4 for local projective measurements—which suits the paradigm of measurement-based computing—the argument obviously holds true for positive operator-valued measurements. The proof is completely analogous, with $\sum_j |\eta_j\rangle\langle\eta_j| = \mathbb{1}$ being replaced by a more general resolution of the identity.

This argument shows that 1D GPEPSs cannot be used as quantum wires even when allowing for arbitrary non-Gaussian local measurements. Note that for this argument to hold, completeness of the measurement bases are indeed necessary: For single outcomes, the condition of the output being up to a constant unitarily equivalent to the input can well be achieved also for matrices having a different structure; but then one cannot assure that this is true for each outcome j of the measurement. This, however, is required to faithfully transport quantum information. If we allow for a finite rate of failure outcomes j in individual steps, then the overall probability of success will asymptotically again become 0 at an exponential rate.

E. Gaussian cluster states under arbitrary encodings and in higher dimensional lattices

One might wonder whether this limitation can be overcome if a large number of physical modes of a higher dimensional lattice are allowed to carry logical information. The same argument, actually, can be applied to a $k \times k \times \cdots \times k \times n$ cubic slab, as a subset of a D-dimensional cubic lattice, where one aims at transporting along the last dimension, with local measurements at each site (Fig. 7). In fact, contracting any dimension except from the last—so summing over all joint indices—one arrives at a GMPS with a bond dimension that is exponential in k. This, however, is a constant. This situation is hence again covered by a GMPS, as long as one allows for more than one physical mode and more than one virtual mode per site. Since the argument in Sec. V B does not make use of the fact that we have only a single virtual and physical mode per site: only that now $|0\rangle^{\otimes(k^{(D-1)})}$ are being fed into the sequential preparation.

Observation 5. Exponential decay of subspace localizable entanglement in a higher dimensional lattice. Let G be a one-dimensional GPEPS, and A and B two sites in a $k \times k \times \cdots \times k \times n$ slab as a subset of a D-dimensional cubic lattice, and denote by i, j the last coordinate of sites A and B. Then

$$E_S(A,B) \leqslant c_4 e^{-d(i,j)/\xi_4}, \tag{52}$$

where $c_4, \xi_4 > 0$ are constants, even if arbitrary local measurements are taken into account.

So even encodings in higher dimensional Gaussian cluster states do not alter the situation that one cannot transport along

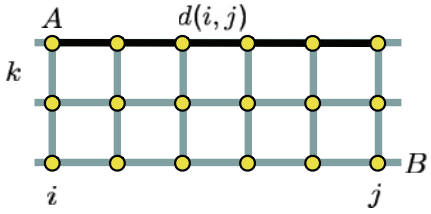

FIG. 7. (Color online) A slab of a $k \times n$ lattice, aiming at using the second dimension as a quantum wire for quantum computation. Again, the probability of transport between A and B decays exponentially with the distance along the last dimension.

a given dimension, if one wants to think of such slabs as perfect primitives being used in a universal quantum computing scheme.

F. Role of error correction and fault tolerance

Observations 2 and 5 show that, under mild conditions, Gaussian cluster states need *not be used as* or *made almost-perfect resources by* local measurements alone. This constitutes a significant challenge for MBQC with Gaussian cluster states but does not rule out this possibility. In this section, we briefly comment on ways that might allow one to overcome the limitations identified here.

Clearly, it is very much conceivable that this observation may again be overcome by concatenated encoding in fault-tolerant schemes, effectively in slabs whose width scales with the length of the computation: Rather, at the level of finite encodings, the resource cannot be uplifted to a perfect resource. The situation encountered here—having pure Gaussian states—hence has some similarity to *noisy finite-dimensional cluster states* built with *imperfect operations* [14,15]. Considering the preparation of the quantum wire and the transport by local measurements as a sequence of teleportations with not fully entangled resources, this means that every step adds a given amount of noise to the quantum information. In finite-dimensional schemes, if this noise corresponds to an error rate below the fault tolerance, a nested encoding with an error correction code allows one to perform computations. The size of the code grows polynomially with the size of the circuit one wishes to implement. In addition to this intrinsic error, any physical implementation will, of course, also suffer from experimental errors which must also be compensated by error correction schemes. Thus, the combined error rate must be below the fault-tolerance threshold. It is therefore possible that recognizing all finite squeezings as full quantum errors—which has to be done in the light of the results of the present work—and using suitable concatenated encodings over polynomially many slabs, there exists a finite squeezing allowing for full universal quantum computation with eventual polynomial overhead. The question whether schemes such as these—or ones where suitable polynomially sized complex structures are "pinched" out of a large lattice—that are universal can be constructed remains a challenging and interesting open question.

G. Ideas on percolation

One possible way forward toward the goal of achieving a fully universal resource under local non-Gaussian measurements is to think of first performing local measurements at each site, aiming at filtering an imperfect qubit, $\mathbb{C}^2$ cluster from a Gaussian cluster state. Ideally, one would arrive at the situation on, say, a cubic lattice of some dimension, where one could extract a *graph state* [46] corresponding to having an edge between nearest neighbors with some finite probability. If this probability p_s is sufficiently high—higher than the appropriate threshold for *edge percolation*—and if one can ensure suitable independence, an asymptotically perfect cluster on a renormalized lattice can be obtained [47–49]. When trying to identify such percolation schemes, one does not have to rely solely on classical percolation schemes, but can also make use of more general repeater-type schemes as in Ref. [50], referred to as *quantum percolation* (see also Ref. [48]). To identify such maps, either classical or quantum, however, appears to be a very challenging task.

One might also ask whether TMSS bonds as such can be transformed into suitable maximally entangled pairs of $\mathbb{C}^2 \otimes \mathbb{C}^2$ systems. This, however, clearly is the case. Again applying a result for finite-dimensional systems to infinite-dimensional ones by making use of appropriate nets of Hilbert spaces, one finds that given a state vector $|\psi_\lambda\rangle$ of a TMSS of some squeezing parameter $\lambda > 0$, the transformation $|\psi_\lambda\rangle$ to $(|0,0\rangle + |1,1\rangle)/\sqrt{2}$ is possible with a generalized local filtering on A only, together with a suitable unitary in B, with a probability of success of [51,52]

$$p = \min[1, 2(1-\lambda^2)]. \tag{53}$$

Hence, whenever $\lambda \geqslant 1/\sqrt{2}$, this transformation can be done deterministically. This has interesting consequences for quantum repeaters. The protocol performing the transformation

$$|\psi_\lambda\rangle_{A,B} \mapsto \frac{1}{\sqrt{2}}(|0,0\rangle + |1,1\rangle) \tag{54}$$

can be implemented by combining A with an ancillary system C, performing a joined unitary transform on A,C, measuring C, and applying another unitary gate on B classically conditioned on the measurement result.

But even if $\lambda < 1/\sqrt{2}$, one can still distill a resource from a collection of TMSSs distributed in a graph, performing an argument involving percolation here. This, however, merely shows that Gaussian states as such can be resources for information processing. Most importantly, this is not the resource anticipated, so not the actual GPEPS, but a collection of suitable TMSS. Thus, non-GPEPS projections cannot be implemented with linear optics without a massive overhead. Finally, the eventually created qubit cluster state would be obtained in a *single-rail* representation where measurements in the superposition bases, which are needed for the actual computation, are experimentally very difficult and require additional photons. So the question of actual universality of the Gaussian cluster state, under all fair meaningful ways of defining a set of rules, remains an interesting and challenging question.

H. Remarks on 1D Gaussian quantum repeaters

We finally briefly reconsider the question of a quantum repeater setting based on general non-Gaussian operations. We have shown that it is not possible to obtain a finitely entangled state for an arbitrary long 1D GPEPS. However, what is also true at the same time is that a sequential repeater scheme based on sufficiently entangled TMSSs *before* the PEPS projection does yield a nondecaying entangled bond between the end points. That is, using only projective local measurements of each of the sites, one can transform a collection of distributed TMSSs in a 1D setting into a maximally entangled qubit pair shared between the end sites. To show this, it suffices to revisit the situation for three sites, as the general statement on N sites follows immediately by iteration.

Now consider the quantum repeater setting and assume for simplicity that we already have a qubit Bell pair, $|\phi\rangle_{A,B_1} = (|0,0\rangle + |1,1\rangle)/\sqrt{2}$, which we want to swap through a TMSS $|\psi_\lambda\rangle_{B_2,C}$ with $\lambda \geqslant 1/\sqrt{2}$. We can use the higher, unoccupied Fock levels of the state vector $|\phi\rangle_{A,B_1}$ as an ancilla to transform $|\psi_\lambda\rangle_{B_2,C}$ according to Eq. (54). As the final unitary on C after local operations and classical communication with one-way classical communication does not change the entanglement, we can also omit it. As the unitary, the ancilla measurement, and the final Bell measurement on B_1, B_2 are equivalent to a single projective measurement on B_1, B_2, it is possible to swap entanglement through a physical TMSS perfectly. Needless to say, this will be a highly non-Gaussian, complicated operation and will not overcome the limitation of Gaussian cluster states discussed.

VI. DISCUSSION AND SUMMARY

In this article, we have assessed the requirements for possible architectures when using Gaussian states as resources for MBQC and for entanglement distribution by means of quantum repeater networks. Using a framework of GPEPSs, we have shown that under Gaussian measurements only, the localizable entanglement decays exponentially with the distance in arbitrary graphs. This rules out the possibility of processing or even transporting quantum information with Gaussian measurements only.

The preceding results also show that Gaussian cluster states—under mild conditions of the encoding of logical information in slabs, rather than general encodings in the entire lattice—cannot be used as or made perfect universal resources for MBQC. No information can be transmitted beyond a certain influence region, and hence, no arbitrarily long computation can be sustained. Now if one allows for a higher energy, and hence larger two-mode squeezing, in the resource states, this influence region will become larger. In other words, small-scale implementations as proof-of-principle experimental realizations of such an idea will be entirely unaffected by this: Any state with finite energy will constitute some approximation of the idealized improper state having infinite energy, and its outcomes in measurements will approximate the idealized ones. However, with this state, one could not go ahead with an arbitrarily long computation. This observation shows that Gaussian cluster states are fine examples of states that eventually allow for the demonstration of the functioning of a CV quantum computer, possibly realized using the many modes available in a *frequency comb* [5–7].

Also, we have discussed the requirements for fault tolerance and quantum error correction for such schemes, yet to be established, in that any finite squeezings essentially have to be considered full errors in a concatenated encoding scheme. This work motivates further studies of the fault tolerance of systems with a finite-dimensional logical encoding in infinite-dimensional systems. But it also strongly suggests that it could be a fruitful enterprise to further at alternative CV schemes not directly involving Gaussian states, but other relatively feasible classes of states, such as coherent superpositions of a few Gaussian states like the so-called cat states, which have turned out to be very useful within another computation paradigm [53]. We hope that this article will contribute to clarifying the requirements that any architecture eventually must meet based on the interesting idea of doing quantum computing by performing local measurements on Gaussian or non-Gaussian states of light.

ACKNOWLEDGMENTS

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APPENDIX: PROOF OF LEMMA 4

Let $A, B \in \mathbb{C}^{2\times 2}$, with $A, B \geqslant 0$. We set

$$c = \frac{\|A^{1/2}B^{1/2}\|^2}{\|A\|\|B\|}. \tag{A1}$$

The inequality $c \leqslant 1$ follows directly from the submultiplicativity of the operator norm, while equality holds if and only if A and B commute. Rewriting

$$\begin{aligned}\lambda_n(A+B) &= \mathrm{tr}(A+B) - \lambda_1(A+B)\\ &= \mathrm{tr}(A+B) - \|A+B\|,\end{aligned} \tag{A2}$$

we can now use a sharpened form of the triangle inequality for the operator norm of 2×2 matrices in Ref. [54] to obtain

$$\begin{aligned}\lambda_2(A+B) &= \mathrm{tr}(A+B) - \|A+B\|\\ &\geqslant \mathrm{tr}(A+B) - \tfrac{1}{2}(\|A\| + \|B\|)\\ &\quad + \tfrac{1}{2}(\|A\| - \|B\|)^2 + 4\|A^{1/2}B^{1/2}\|^2)^{1/2}.\end{aligned} \tag{A3}$$

If now $c < 1$, then there exists a $\delta > 0$ such that

$$\begin{aligned}\lambda_2(A+B) &\geqslant \mathrm{tr}(A+B) - \big((\|A\| - \|B\|)^2\\ &\quad + 4\|A\|\|B\|\big)^{1/2} + \delta\\ &= \mathrm{tr}(A+B) - (\|A\| + \|B\|) + \delta \qquad \text{(A4)}\\ &= \lambda_2(A) + \lambda_2(B) + \delta, \qquad \text{(A5)}\end{aligned}$$

which proves Lemma 4.

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