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Entanglement and communication-reducing properties of noisy N-qubit states

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We consider properties of states of many qubits, which arise after sending certain entangled states via various noisy channels (white noise, colored noise, local depolarization, dephasing, and amplitude damping). Entanglement of these states and their ability to violate certain classes of Bell inequalities are studied. States which violate them allow a higher than classical efficiency in solving related distributed computational tasks with constrained communication. This is a direct property of such states—not requiring their further modification via stochastic local operations and classical communication such as entanglement purification or distillation procedures. We identify families of multiparticle states which are entangled but nevertheless allow the local realistic description of specific Bell experiments. For some of them, the "gap" between the critical values for entanglement and violation of Bell inequality remains finite even in the limit of infinitely many qubits.

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I. INTRODUCTION

Despite the considerable progress made in understanding entanglement, the question of whether every entangled state does not admit a local realistic simulation is as yet unanswered. Bell has shown that certain pure entangled states violate constraints imposed by local hidden variable models [1]. Bell's result was generalized by Gisin and Peres who demonstrated the violation for all bipartite pure entangled states [2,3]. Popescu and Rohrlich showed that no local realistic description is possible for any pure multipartite entangled state; the proof involved post-selection [4]. Without post-selection, it is not clear whether there are pure entangled states which admit a local realistic model for all possible measurements. A Bell experiment with two settings per observer, in which only correlation functions were measured, indeed admits a local hidden-variable explanation even for some pure entangled states [5,6].

For mixed states, this relation is even subtler. Werner states are an example of bipartite entangled mixed states which allow a local realistic model for direct measurements [7,8]. Almeida *et al.* found recently that the range of entanglement admixture for which the state of two d-level systems is entangled and also admits a local hidden-variable model for all measurements decreases proportionally to $\log(d)/d$ [9]. Also, some genuinely tripartite entangled mixed states can admit a hidden-variable description for all measurements [10]. It was shown that entangled states upon sequential local measurements may be transformed into ones that do not allow a local realistic description [11–15]. Note, however, that this is not a *direct* property of such states, only the final states which result out of such transformations are endowed with it. The relation between entanglement and local realism for multipartite mixed states is still largely unexplored. Our work addresses this problem.

This relation is of importance not only to fundamental research but also in the context of quantum communication and quantum computation. For certain tasks, such as quantum communication complexity problems [16,17] or device-independent quantum key distribution [18,19], entangled states are useful only to the extent that they violate Bell inequalities. Furthermore, entangled states which violate certain Bell inequalities but satisfy others are useful for particular quantum communication complexity problems directly related to the violated inequalities (for details of the link between the inequalities and communication complexity problems, see [17]).

In such problems, several partners have disjoint sets of data and under a strict communication constraint, e.g., to one bit per partner, are asked to give the value of a task function which depends on all data. The amount of violation of a Bell inequality for correlation functions related to the problem is proportional to the increase of the probability to get the correct value of the task function, which quantum protocols involving entangled states allow in contrast to the optimal classical protocol. Note that often additional post-processing of experimental data requires additional classical communication and therefore increases communication complexity of quantum protocols. Therefore, states which violate certain Bell inequalities after sequential measurements or post-selection are usually less efficient in terms of communication complexity reduction than the states which violate the inequalities directly. It is thus important to make both classifications of entangled states: into admitting and not admitting local realistic models, and into violating and not violating a given Bell inequality.

Entanglement and Bell violation of different noisy states have already been studied by several authors [20–25]. All this indicates that entanglement and the impossibility of a local hidden variable model are not only different concepts, but also truly different resources [26]. Our aim is to identify a class of states that demonstrates this difference in a striking way.

We consider states of N two-level systems resulting from sending different entangled states via noisy channels. Noisy states are of special importance, because they take into account errors inevitable in any laboratory. More specifically, we study white-noise admixture which is often used to model imperfections of setups involving a single crystal in which spontaneous parametric down-conversion takes place. We consider also colored noise admixture which was shown to be appropriate, e.g., in describing states generated in multiple entanglement swapping [27]. Typical noisy channels (depolarization, dephasing, amplitude damping) which act independently on every qubit are also studied. They find applications in modeling random environment and dissipative processes [28].

The treatment of noise is static here, as the sole purpose of the paper is to address properties of states. Especially, we aim to find exact values of the "gaps" (originally found by Werner [7] for simpler cases) between admixtures of a broader class of noise than usually studied. Such a gap is the difference between the values of the critical admixture of a specific (toy) model noise: the value which is enough to kill nonclassical correlations, and the value that kills entanglement present in the original state. Thus the problems we study are independent of the dynamics, and concern solely properties of states. Nevertheless, it is an interesting question to study such gaps within dynamical models of noise, because for specific experimental configurations one can expect, due to environment-system interactions, or dephasing, specific types of noises to emerge. This problem is left open for a later study.

Here we find states for which even an infinitesimally small admixture of infinitesimal weak entangled state results in a nonseparable state, while to violate standard Bell inequalities (with two settings per party) the admixture has to scale at least as $1/\sqrt{d}$, where d is the dimension of N qubits, i.e., $d = 2^N$. This shows a remarkable "gap" between the critical parameters for entanglement and those for violation of standard Bell inequalities. We observe that keeping the same amount of noise but changing the type of noise drastically changes entanglement and communication-reducing properties of the states, i.e., whether the states allow a higher than classical reduction of communication complexity. Furthermore, we find mixed states for which this gap remains finite even in the limit of infinitely many qubits.

II. TOOLBOX

Our tools consist of entanglement criteria in terms of correlation functions [29], which will prove handy for comparison with conditions for violation of Bell inequalities. We shall take into account sets of Bell inequalities for two and more measurement settings [30–35]. We now describe these tools in more detail.

Arbitrary state of many qubits can be decomposed into

$$\rho = \frac{1}{2^N} \sum_{\mu_1,\ldots,\mu_N=0}^{3} T_{\mu_1\ldots\mu_N} \sigma_{\mu_1} \otimes \cdots \otimes \sigma_{\mu_N}, \tag{1}$$

where $\sigma_{\mu_n} \in \{\mathbb{1}, \sigma_x, \sigma_y, \sigma_z\}$ is the μ_nth local Pauli operator of the nth party ($\sigma_0 = \mathbb{1}$), and $T_{\mu_1\ldots\mu_N} \in [-1, 1]$ are the components of the (real) extended correlation tensor $\hat{T}$. They are the expectation values $T_{\mu_1\ldots\mu_N} = \mathrm{Tr}[\rho(\sigma_{\mu_1} \otimes \cdots \otimes \sigma_{\mu_N})]$. Thus, description in terms of correlation tensor is equivalent to description in terms of density operator. Fully separable states are endowed with fully separable extended correlation tensor, $\hat{T}^{\mathrm{sep}} = \sum_i p_i \hat{T}_i^{\mathrm{prod}}$, where $\hat{T}_i^{\mathrm{prod}} = \hat{T}_i^{(1)} \otimes \cdots \otimes \hat{T}_i^{(N)}$, and each $\hat{T}_i^{(n)}$ describes a pure one-qubit state. A state ρ, with correlation tensor $\hat{T}$, is entangled if there exists a G such that [29]

$$\max_{\hat{T}^{\mathrm{prod}}} (\hat{T}, \hat{T}^{\mathrm{prod}})_G < (\hat{T}, \hat{T})_G = ||\hat{T}||_G^2, \tag{2}$$

where maximization is taken over all product states and $(.,.)_G$ denotes a generalized scalar product, with a positive semidefinite metric G. We focus on diagonal G's, for which the scalar product is given by

$$(\hat{T}, \hat{T}')_G = \sum_{\mu_1,\ldots,\mu_N=0}^{3} T_{\mu_1\ldots\mu_N} G_{\mu_1\ldots\mu_N} T'_{\mu_1\ldots\mu_N}. \tag{3}$$

The criterion is valid also when the sums of (3) run through the values $j_n = 1, 2, 3$, which will be often referred to as x, y, z.

We compare this entanglement criterion with criteria for violation of Bell inequalities. It was shown that a simple sufficient condition for existence of a local realistic description of the correlation functions obtained in any Bell experiment with two measurement settings per observer has the following form [32]:

$$\mathcal{C} \equiv \max \sum_{j_1,\ldots,j_N=1}^{2} T_{j_1\ldots j_N}^2 \leqslant 1, \tag{4}$$

where maximization is taken over all possible independent choices of local planes in which the two settings lie. This condition is necessary and sufficient in the case of two qubits [36].

We shall also use another necessary and sufficient condition, this time for violation of a set of tight Bell inequalities with many measurement settings per observer [35]. For the case of $N+1$ observers, all of which but the last one choose between four settings, and the last one between two settings, this condition reads

$$\mathcal{D} \equiv \max \sum_{j_1(k),\ldots,j_N(k)=1}^{2} \sum_{k=1}^{2} T_{j_1(k)\ldots j_N(k)k}^2 \leqslant 1, \tag{5}$$

where maximization is over all possible independent choices of local Cartesian frame basis vectors used by the observers to fix the measurement directions determining the correlation tensor components. That is, we allow each observer to define its triad of orthogonal basis directions, which define the correlation tensor components. This condition is more demanding than (4) because the coordinate systems denoted by the indices $j_1(1),\ldots, j_N(1)$ do not have to be the same as $j_1(2),\ldots, j_N(2)$. It is necessary for the existence of a local realistic model [35], or equivalently, its violation is sufficient for the nonexistence of such models.

III. NOISES

The states to be studied here are of two general types. (i) Mixtures of an entangled state ρ and white or colored noise ρ_{noise}:

$$\rho(\Upsilon) = \Upsilon\rho + (1-\Upsilon)\rho_{\mathrm{noise}}, \tag{6}$$

where Υ is the entanglement admixture. (ii) States arising from local noisy channels [28], i.e., of the form

$$(\mathcal{E} \otimes \cdots \otimes \mathcal{E})(\rho) = \frac{1}{2^N} \sum_{\mu_1,\ldots,\mu_N=0}^{3} T_{\mu_1\ldots\mu_N} \mathcal{E}(\sigma_{\mu_1}) \otimes \cdots \otimes \mathcal{E}(\sigma_{\mu_N}), \tag{7}$$

where $\mathcal{E}$ is a map describing depolarization, dephasing, or amplitude damping of a single qubit. According to (7), such noises are fully described by their action on local Pauli operators.

Each type of noise considered is parametrized with a single variable (it is either entanglement admixture Υ or the strength of local decoherence), and therefore the resulting states are also characterized by this variable. We choose the parameters such that value '1' corresponds to no noise, whereas value '0' corresponds to total noise which immediately destroys all initial entanglement. Using the described separability criterion, we determine the threshold value of the parameter, above which the resulting state is entangled. Next, using conditions (4) and (5) for the described Bell inequalities, we find the maximal parameter below which the state does not violate them. Finally, we contrast these two critical values. We summarize our results in Table II.

A. White noise

White noise is represented by a totally mixed state $\rho_{\text{noise}} = \frac{1}{2^N}\mathbb{1}$, where N gives the number of qubits. A channel introducing the white noise to a system is the globally depolarizing channel

$$\mathcal{E}_\Upsilon(\rho) = \Upsilon\rho + (1-\Upsilon)\tfrac{1}{2^N}\mathbb{1}. \tag{8}$$

Therefore, the correlation tensor of the state after the globally depolarizing channel, $\hat{T}'$, is related to the initial state by the admixture parameter $\hat{T}' = \Upsilon\hat{T}$. The operator-sum representation

$$\mathcal{E}_\Upsilon(\rho) = \Upsilon\rho + \frac{1-\Upsilon}{2^{2N}} \times \sum_{\mu_1,\ldots,\mu_N=0}^{3} \sigma_{\mu_1} \otimes \cdots \otimes \sigma_{\mu_N} \rho \sigma_{\mu_1} \otimes \cdots \otimes \sigma_{\mu_N} \tag{9}$$

reveals that a white-noise admixture acts in a correlated way on all the qubits.

B. Colored noise

We will consider colored noise represented by a product state $\rho_{\text{noise}} = |0\rangle\langle 0| \otimes \cdots \otimes |0\rangle\langle 0|$, where $|0\rangle$ is the eigenstate of the local σ_z Pauli operator. Such a noise brings perfect correlations along z directions to the system.

C. Local depolarization

In many cases, noise affects independently every qubit. For example, local depolarization can be caused by a random environment acting autonomously on each subsystem. The local depolarization is defined for a single qubit in the familiar way:

$$\mathcal{E}_p(\rho) = p\rho + (1-p)\tfrac{1}{2}\mathbb{1}, \tag{10}$$

i.e., it mixes the local state with the white noise, where p describes the fraction of initial state still present after the decoherence. To see the effect of local depolarization on many qubits, we find its effect on local Pauli operators

$$\begin{aligned} \mathcal{E}_p(\mathbb{1}) &= \mathbb{1}, \quad \mathcal{E}_p(\sigma_x) = p\sigma_x, \\ \mathcal{E}_p(\sigma_y) &= p\sigma_y, \quad \mathcal{E}_p(\sigma_z) = p\sigma_z, \end{aligned} \tag{11}$$

and follow formula (7).

In general, the critical values arising from local depolarization and white noise can be different. However, in our case the critical parameters turn out to be the same because of the structure of violation conditions for the Bell inequalities and the form of the entanglement criterion we use. Since these conditions involve only N-party correlations, local depolarization introduces a factor of p^N to the elements of correlation tensor entering them, while white noise admixed to the system introduces a factor of Υ. Therefore, the critical values obtained using p^N and Υ are equal, $p_{\text{cr}}^N = \Upsilon_{\text{cr}}$, independently of the state for which they are computed (the numerical value can, of course, vary from state to state).

D. Dephasing

Local depolarization describes gradual loss of coherence in all bases. It may happen that coherence is lost in a preferred basis. This type of noise is described by a dephasing channel defined by its action on local Pauli operators:

$$\begin{aligned} \mathcal{E}_\lambda(\mathbb{1}) &= \mathbb{1}, \quad \mathcal{E}_\lambda(\sigma_x) = \sqrt{\lambda}\sigma_x, \\ \mathcal{E}_\lambda(\sigma_y) &= \sqrt{\lambda}\sigma_y, \quad \mathcal{E}_\lambda(\sigma_z) = \sigma_z, \end{aligned} \tag{12}$$

where the σ_z basis is chosen to be preferred by decoherence, and λ describes its strength. Clearly, for $\lambda = 1$, the initial state is unchanged; and for $\lambda = 0$, the final state has only classical correlations along local z directions.

E. Amplitude damping

An amplitude-damping channel is used to describe energy dissipation from a quantum system. Under amplitude damping, a system has a finite probability γ to lose an excitation. In terms of local Pauli operators, this channel is descried as

$$\begin{aligned} \mathcal{E}_\gamma(\mathbb{1}) &= \mathbb{1} + (1-\gamma)\sigma_z, \quad \mathcal{E}_\gamma(\sigma_x) = \sqrt{\gamma}\sigma_x, \\ \mathcal{E}_\gamma(\sigma_y) &= \sqrt{\gamma}\sigma_y, \quad \mathcal{E}_\gamma(\sigma_z) = \gamma\sigma_z. \end{aligned} \tag{13}$$

Note that the components of the correlation tensor of a state after amplitude damping which contain the z indices are given by the sums of the initial correlation tensor components with both z indices and zero indices, e.g.,

$$T'_{\underbrace{z\ldots z}_{k}0\ldots 0} = \sum_{l_1,\ldots,l_k=\{0,3\}} T_{l_1\ldots l_k 0\ldots 0}(1-\gamma)^{n_0}\gamma^{n_3} \tag{14}$$

where $n_0 \equiv \sum_{j=1}^{k} \delta_{l_j,0}$ gives the number of indices $l_1, \ldots, l_k$ equal to 0, and similarly $n_3 \equiv \sum_{j=1}^{k} \delta_{l_j,3} = k - n_0$ denotes the number of indices equal to 3.

Having described the noises of interest, we move to studies of their influence on certain classes of initially entangled states.

IV. NOISY STATES

We begin with the Bell state of two qubits and mix it with white and colored noise. The state with white noise is the Werner state known to admit a local hidden-variable model for certain admixtures despite being entangled. Interestingly, the states with colored noise, which are maximally entangled mixed states [37,38], will be shown to be entangled and to not violate standard Bell inequalities in an even bigger range of mixing. We then show similar results for the GHZ states and generalized GHZ states. For some of them, the critical admixture of entanglement below which the state admits a local hidden-variable model scales polynomially with dimension of the system, and the mixed state is entangled already for an infinitesimally small admixture of infinitesimal entanglement.

Next, we discuss noisy states arising from independent local decoherence. We start with generalized GHZ states as initial states and show that even in the limit of infinitely many qubits there is still a finite gap between the critical parameter for entanglement and the one for violation of Bell inequalities. We show similar results when the initial state is a W state. Roughly speaking, a simple application of the entanglement criterion detects entanglement at least quadratically better than the Bell inequalities; i.e., the critical value for entanglement is at most equal to the square of the critical value to satisfy the Bell inequalities.

A. Bell state

1. White noise

We first rederive known results for a Werner state of two qubits with our tools. It is a mixture of a maximally entangled state $\rho = |\phi^+\rangle\langle\phi^+|$ and white noise $\rho_{\text{noise}} = \frac{1}{4}\mathbb{1}$, where $|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ and $|0\rangle$ ($|1\rangle$) denotes the eigenstate of σ_z operator with eigenvalue $+1$ (-1). The family of Werner states is an archetypical example of a state set which contains states that do not violate Bell inequalities despite being entangled.

Since the white-noise state exhibits no correlations, the correlation tensor components $T'_{j_1 j_2}$ of the Werner state are related to the components $T_{j_1 j_2}$ of $|\phi^+\rangle$ by the admixture factor, $T'_{j_1 j_2} = \Upsilon T_{j_1 j_2}$. The only nonvanishing correlation tensor elements of maximally entangled states lie on the diagonal and are equal to ± 1 (this is so when the two-particle correlation tensor is put in a Schmidt form). If one chooses to sum over $j_n = 1, 2, 3$ in the scalar products of criterion (2), the left-hand side is given by the maximal Schmidt coefficient of the correlation tensor. For the Werner state it equals Υ. The right-hand side reads $3\Upsilon^2$. Thus, the criterion reveals entanglement for all the states of the family, i.e., for $\Upsilon_{\text{ent}} > \frac{1}{3}$. On the other hand, the necessary and sufficient condition for a local realistic model, in the case of a standard two-settings-per-partner Bell experiment (4), is satisfied for $\Upsilon_{\text{lr}} \leqslant \frac{1}{\sqrt{2}}$. Thus, for a considerable range of $\Upsilon \in (\frac{1}{3}, \frac{1}{\sqrt{2}}]$ the state is entangled, nevertheless Bell experiments involving standard inequalities have a local realistic explanation. One could call this range of Υ a "Werner gap."

TABLE I. Critical value of entanglement admixture above which the two-qubit state $\Upsilon|\phi_+\rangle\langle\phi_+| + (1-\Upsilon)\rho_{\text{noise}}$ is entangled (middle column) and allows reduction of communication complexity, i.e., violates standard Bell inequalities (right column), for different types of noise (left column). In the left column, $\mathbb{1}\otimes\mathbb{1}$ denotes the white noise and, e.g., $|\pm\rangle_{kk}\langle\pm| \otimes |\mp\rangle_{ll}\langle\mp|$ denotes the colored noise which is a product state of either $|+\rangle_{kk}\langle+| \otimes |-\rangle_{ll}\langle-|$ or $|-\rangle_{kk}\langle-| \otimes |+\rangle_{ll}\langle+|$, where $|\pm\rangle_k$ is the eigenstate of the Pauli σ_k operator with eigenvalue ± 1 (either the upper or lower signs enter the states of the noise).

Type of noise	Entanglement	Comm. reduction
$\mathbb{1}\otimes\mathbb{1}$	$\Upsilon > \frac{1}{3}$	$\Upsilon > \frac{1}{\sqrt{2}} = 0.70711$
$|\pm\rangle_{zz}\langle\pm| \otimes |\pm\rangle_{zz}\langle\pm|$ $|\pm\rangle_{yy}\langle\pm| \otimes |\mp\rangle_{yy}\langle\mp|$ $|\pm\rangle_{xx}\langle\pm| \otimes |\pm\rangle_{xx}\langle\pm|$	$\Upsilon > 0$	$\Upsilon > 0$
$|\pm\rangle_{zz}\langle\pm| \otimes |\mp\rangle_{zz}\langle\mp|$ $|\pm\rangle_{yy}\langle\pm| \otimes |\pm\rangle_{yy}\langle\pm|$ $|\pm\rangle_{xx}\langle\pm| \otimes |\mp\rangle_{xx}\langle\mp|$	$\Upsilon > 0$	$\Upsilon > \frac{1}{\sqrt{2}} = 0.70711$
$|\pm\rangle_{xx}\langle\pm| \otimes |\pm\rangle_{yy}\langle\pm|$ $|\pm\rangle_{xx}\langle\pm| \otimes |\mp\rangle_{yy}\langle\mp|$ $|\pm\rangle_{xx}\langle\pm| \otimes |\pm\rangle_{zz}\langle\pm|$ $|\pm\rangle_{xx}\langle\pm| \otimes |\mp\rangle_{zz}\langle\mp|$ $|\pm\rangle_{yy}\langle\pm| \otimes |\pm\rangle_{zz}\langle\pm|$ $|\pm\rangle_{yy}\langle\pm| \otimes |\mp\rangle_{zz}\langle\mp|$	$\Upsilon > 0$	$\Upsilon > 0.56731$

2. Colored noise

Interestingly, changing the type of noise from white to colored influences both entanglement of the state and possibility of local realistic model. We have investigated critical admixtures of different types of noise above which condition (4) is satisfied and summarize them in Table I. Changing the type of colored noise alone, although it does not change the entanglement threshold of the state, dramatically influences its communication-reducing properties. The splitting of this table into different rows is motivated by different relations between correlations present in the noise and in the entangled state $|\phi^+\rangle$. In the first row, the white noise has no correlations. In the second row, the noises have some of the correlations of the $|\phi^+\rangle$ state. Therefore, for all $\Upsilon > 0$ there are perfect correlations in the system (in the basis of states of noise) and additionally at least some correlations in a complementary measurement direction. This explains the violation of a two-setting Bell inequality [39]. In the third row, the noise has exactly opposite correlations to those present in the $|\phi^+\rangle$ state. In the last row, the noises have correlations of a different character than those of the entangled state.

The Werner states are not the ones with the largest possible gap. For example, if one admixes, e.g., colored noise $\rho_{\text{noise}} = |\pm\rangle_{zz}\langle\pm| \otimes |\mp\rangle_{zz}\langle\mp|$ to the $|\phi^+\rangle$ Bell state, the resulting state is entangled already for an infinitesimally small value of Υ, and it satisfies condition (4) for all $\Upsilon_{\text{lr}} \leqslant \frac{1}{\sqrt{2}}$. Thus, the range of Υ for which the state admits local realistic model for standard correlation Bell experiments and is still entangled is much larger than for the Werner state. Moreover, this is the maximal possible range (there is no other state or dichotomic measurements which would give a bigger Werner gap), because the critical value $\Upsilon_{\text{lr}} = \frac{1}{\sqrt{2}}$ corresponds to the maximal violation of local realism [40]. We note that such mixed states are known to be maximally entangled [37,38].

B. GHZ state

1. White noise

The presented tools allow us to construct and investigate entangled states of multiple qubits, with a nonzero Werner gap, in a systematic way. We first consider the Werner-like states of N qubits which are mixtures of the GHZ state $|\mathrm{GHZ}_N\rangle = \frac{1}{\sqrt{2}}(|0\ldots0\rangle + |1\ldots1\rangle)$ and the white noise. Using criterion (2), one finds $\Upsilon_{\mathrm{ent}} = 1/(2^{N-1}+1)$ for the critical admixture above which the state is entangled [29,41]. The critical value for violation of a complete set of standard Bell inequalities for correlation functions equals $\Upsilon_{\mathrm{lr}} = 1/\sqrt{2^{N-1}}$ (see [32]). Therefore, for $\Upsilon \in (\frac{1}{2^{N-1}+1}, \frac{1}{\sqrt{2^{N-1}}}]$ the state is entangled, but all two-setting correlation Bell experiments admit a local realistic model. Also the multiple-setting inequalities of Ref. [35] are all satisfied in this range.

To illustrate how the range of the Werner gap can depend on the Bell inequality, we consider the inequalities of Refs. [42,43]. If one considers all possible settings, restricted to one measurement plane on the Bloch sphere for each observer, the critical value for violation of local realism changes to $\Upsilon_{\mathrm{lr}}^{\infty} = 2(2/\pi)^N$, see [42], and therefore decreases the Werner gap. This result is a limiting case for inequalities involving M settings per party studied in [43]. These inequalities involve measurement settings (again in a specific plane for each observer) evenly spaced at the Bloch sphere. One has $\Upsilon_{\mathrm{lr}}^{\infty} = \lim_{M\to\infty} \Upsilon_{\mathrm{lr}}^{M}$ [43], notation is obvious here. One may ask for how many settings is the critical entanglement admixture for violation of local realism for a finite and continuum number of settings already very close. For bigger M, one finds using Taylor series that $\Upsilon_{\mathrm{lr}}^{M} = \Upsilon_{\mathrm{lr}}^{\infty}[1 + \frac{\pi^2}{24}\frac{N-3}{M^2} + O(\frac{N^2}{M^4})]$. If one neglects all the small terms of $O(\frac{N^2}{M^4})$, the relative error $\epsilon = (\Upsilon_{\mathrm{lr}}^{M} - \Upsilon_{\mathrm{lr}}^{\infty})/\Upsilon_{\mathrm{lr}}^{\infty}$ is given by $\epsilon \approx 4\pi^2\frac{N-3}{M^2}\%$. Thus, for $M = N$ the two critical admixtures are close even for a few particles (ϵ smaller than 4% for all $N \geqslant 4$).

2. Colored noise

Similar to the case of the Bell states, the range of the Werner gap for GHZ states also depends on the type of admixed noise. For odd-N GHZ states, the correlations $T_{z\ldots z}$ vanish, and it is interesting to consider the colored noise $\rho_{\mathrm{noise}} = |0\rangle\langle 0|^{\otimes N}$ which reintroduces the missing correlations. The full correlation tensor of $\rho(\Upsilon)$, i.e., the one covering "Greek" indices from 0 to 3, has the following nonvanishing components: $T_{z\ldots z} = 1 - \Upsilon$, and also 2^{N-1} components with $2k$ indices equal to y and the remaining indices equal to x (where $k = 0, 1, \ldots, \frac{N-1}{2}$). These latter ones are given by $(-1)^k\Upsilon$. Finally, one finds $2^{N-1} - 1$ components with $2k$ indices (where $k = 1, \ldots, \frac{N-1}{2}$) set at 0 and the remaining indices set to z. All these have the value of 1. Consider a metric G with only the following nonzero elements: $G_{zz0\ldots0} = \Upsilon$ and $G_{i_1\ldots i_N} = 1$ for the components with $2k$ indices equal to y and the rest equal to x. For such a metric, the maximum of the scalar product on the left-hand side of condition (2) is equal to Υ. The right-hand side equals $||\hat{T}||_G^2 = \Upsilon + 2^{N-1}\Upsilon^2$, which is always greater than Υ. Thus, the state is entangled already for an infinitesimally small Υ.

To investigate the direct communication-reducing properties of the state, we employ condition (4). Depending on the choice of the observation plane, the left-hand side of Eq. (4) reads $2^{N-1}\Upsilon^2$ for the xy plane; $(1-\Upsilon)^2 + \Upsilon^2$ for the xz plane; and $(1-\Upsilon)^2$ for the yz plane. There is no other plane in which the values would be higher, as the correlation tensor is in its generalized Schmidt form [44,45]. The sum over the settings in the xy plane is greater than the sum over the xz plane for $\Upsilon > 1/(1 + \sqrt{2^{N-1} - 1})$. Thus, for the state $\rho(\Upsilon)$ the left-hand side of (4) is given by

$$\mathcal{C} = \begin{cases} 2^{N-1}\Upsilon^2 & \text{for } \Upsilon \leqslant \frac{1}{1+\sqrt{2^{N-1}-1}}, \\ (1-\Upsilon)^2 + \Upsilon^2 & \text{for } \Upsilon > \frac{1}{1+\sqrt{2^{N-1}-1}}. \end{cases} \tag{15}$$

Therefore, there exists a local realistic model for the correlations obtained in any two-setting correlation Bell experiment if $\Upsilon \leqslant \Upsilon_{\mathrm{lr}} = 1/\sqrt{2^{N-1}}$, which is the same critical value as for the state with white noise (in full analogy to the case of the Bell state). Finally, for $\Upsilon \in (0, \Upsilon_{\mathrm{lr}}]$, entangled state $\rho(\Upsilon)$ admits local realistic description for such Bell experiments. Since the dimension of the system is $d = 2^N$, the range of the Werner gap scales polynomially as $d^{-\frac{1}{2}}$. This is exponentially better than in [9] where the range of the Werner gap scales logarithmically as $\log(d)/d$. However, the model of [9] works for an arbitrary number of settings, whereas here we have studied only two-setting Bell inequalities for correlation functions. Already for the multiple-setting Bell inequalities for correlation functions [35], the range of the corresponding Werner gap is smaller. The left-hand side of (5) is given by $\mathcal{D} = 2^{N-1}\Upsilon^2$ for $N = 3$, and $\mathcal{D} = (1-\Upsilon)^2 + 2^{N-2}\Upsilon^2$ for $N \geqslant 5$. This is illustrated in Fig. 1, where we show the critical entanglement admixtures below which the state satisfies the inequalities. Note that in this case the fact that the condition is satisfied does not guarantee the existence of the local realistic model, because this set of inequalities is not necessary and sufficient for the existence of such a model [35]. We also checked that for the colored noise, the inequalities with continuous settings [42] do not improve the critical admixture for violation of local realism over the multisetting inequalities [35], except for $N = 3$.

C. Generalized GHZ states

1. Colored noise

We give an explicit example of a noisy separable state for which even an infinitesimally small admixture of an infinitesimal weak entangled state results in a nonseparable state. Consider the generalized GHZ state [5,6,35]

$$|\mathrm{GHZ}(\alpha)\rangle = \cos\alpha|0\ldots0\rangle + \sin\alpha|1\ldots1\rangle. \tag{16}$$

It has the following nonvanishing components of the correlation tensor:

$$T_{\underbrace{y\ldots y}_{2k}x\ldots x} = (-1)^k \sin 2\alpha, \quad k = 0, 1, \ldots, \lfloor \tfrac{N-1}{2} \rfloor,$$
$$T_{\underbrace{z\ldots z}_{k}0\ldots0} = \begin{cases} 1 & \text{for} \quad k \text{ even}, \\ \cos 2\alpha & \text{for} \quad k \text{ odd}, \end{cases} \tag{17}$$

and similarly for all permutations of indices.

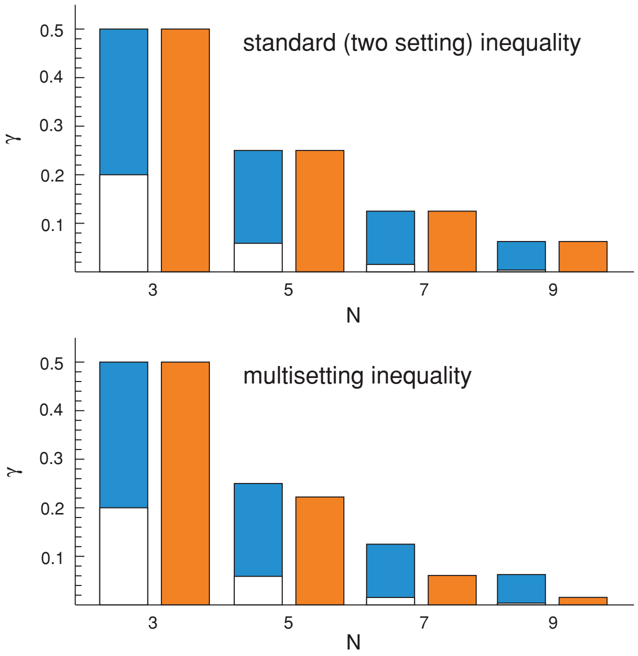

FIG. 1. (Color online) Entanglement and violation of the Bell inequalities. The state $\Upsilon|\mathrm{GHZ}_N\rangle\langle\mathrm{GHZ}_N| + (1-\Upsilon)\rho_{\mathrm{noise}}$ violates the corresponding Bell inequality for values of Υ above the bars. If the value of Υ lies within the blue or orange piece of the bar, the state is entangled but does not violate the Bell inequality and therefore does not allow communication complexity reduction. For each number of qubits, N, we present the results for the white-noise admixture (left bar – blue) and the colored noise $|0\rangle\langle 0|^{\otimes N}$ admixture (right bar – orange). For the white noise, more settings do not lower the critical admixture. For the colored noise, the critical admixture is lowered.

We mix this state with a colored noise $\rho_{\mathrm{noise}} = |0\rangle\langle 0|^{\otimes N}$. If the number of qubits is *even*, this state is entangled and violates Bell inequalities for both infinitesimal α and Υ. This follows from the fact that the state has perfect correlations $T_{z\ldots z} = 1$ and additional correlations in complementary directions, e.g., $T_{x\ldots x} = \Upsilon \sin 2\alpha$. Therefore, summing up the correlations in the xz plane and using the multisetting condition (5) proves the violation.

For an *odd* number of qubits, the range of the Werner gap again scales (independently of α) polynomially with d. First, we show that the mixed state is entangled already for infinitesimal α and Υ, irrespective of the number of qubits. To this aim, take two nonvanishing metric elements $G_{zz0\ldots 0}$ and $G_{x\ldots x}$ to be equal to 1. For this choice, the right-hand side of (2) equals $R = 1 + \Upsilon^2 \sin^2 2\alpha$. The left-hand side of the condition now reads $L = \max(\Upsilon \sin 2\alpha T_x^{(1)} \ldots T_x^{(N)} + T_z^{(1)} T_z^{(2)})$, where we maximize over the choice of local tensors (vectors) $\hat{T}^{(n)}$. We set $T_x^{(n)}$ to the maximal value of 1 for all the parties $n > 2$, and write the tensor elements for the remaining two parties in polar coordinates, $L = \max_{\theta_1,\theta_2}(\Upsilon \sin 2\alpha \sin\theta_1 \sin\theta_2 + \cos\theta_1 \cos\theta_2)$. Since $\Upsilon \sin 2\alpha \leqslant 1$, we have $L \leqslant \max_{\theta_1,\theta_2} \cos(\theta_1 - \theta_2) \leqslant 1$. The maximum is equal to 1 and for all allowed $\alpha > 0$ and $\Upsilon > 0$ it is smaller than the right-hand side. The state is entangled.

For violation of Bell inequalities, consider summation over the xy plane in the necessary and sufficient condition (5). For the present state, it involves $\sum_{k=0}^{(N-1)/2} \binom{N}{2k} = 2^{N-1}$ terms, each equal to $\Upsilon^2 \sin^2 2\alpha$, and gives the critical value of $\Upsilon_{\mathrm{lr}} = \frac{\sqrt{2}}{\sin 2\alpha} 2^{-N/2}$. Therefore the gap $|\Upsilon_{\mathrm{lr}} - \Upsilon_{\mathrm{ent}}|$ scales polynomially with dimension as $1/\sqrt{d}$ for $d = 2^N$.

2. Local depolarization

The nonvanishing correlation tensor elements, after local depolarizing channels are applied to the generalized GHZ state, read

$$T_{\underbrace{y\ldots y}_{2k} x\ldots x} = (-1)^k p^N \sin 2\alpha, \quad k = 0, 1, \ldots, \lfloor \tfrac{N-1}{2} \rfloor$$
$$T_{\underbrace{z\ldots z}_{k} 0\ldots 0} = \begin{cases} p^k & \text{for} \quad k \text{ even}, \\ p^k \cos 2\alpha & \text{for} \quad k \text{ odd}. \end{cases} \tag{18}$$

To show the Werner gap for $N \to \infty$, we first prove that the state is entangled for all $p > \frac{1}{2}$. Choose the following nonzero elements of the metric: $G_{j_1\ldots j_N} = 1$ for $j_n = 1, 2$, $G_{z\ldots z} = 1$ for N even, and $G_{z\ldots z0} = 1$ for odd N. The right-hand side of the entanglement condition (2) is $R = p^{2(N-1+N_2)} + 2^{N-1} p^{2(N-1+N_2)} \sin^2 2\alpha$, where $N_2 = N \bmod 2$ encodes the cases of odd and even N. The left-hand side is maximized if all local tensors are the same and along the z axes and has the value of $L = p^{N-1+N_2}$. Therefore, the state is entangled if $1 < p^{N-1+N_2} + 2^{N-1} p^{N-1+N_2} \sin^2 2\alpha$. To unify the cases of odd and even N, we bound the right-hand side from below using $p^N \leqslant p^{N-1+N_2}$, and obtain the sufficient condition for entanglement. The corresponding critical value is

$$p_{\mathrm{ent}} = (1 + 2^{N-1} \sin^2 2\alpha)^{-\frac{1}{N}} \to \tfrac{1}{2}, \tag{19}$$

where the limit is for $N \to \infty$. Arrows in the following formulas always denote this limit.

The multisetting Bell inequalities give better results than standard inequalities for this state. Consider the violation condition (5) in which the last index of the correlation tensor takes on the values $\{y, z\}$, whereas indices of other parties are either $\{x, y\}$ if the last index is y, or z if the last index is z. (Note that we explicitly make use here of the advantage of the multisetting condition over the two-setting condition). The value of parameter $\mathcal{D}$ is at least (it is higher for N even) equal to $p^{2N}(\cos^2 2\alpha + 2^{N-2} \sin^2 2\alpha)$, where 2^{N-2} gives the number of nonzero correlation tensor elements with y index at the last position. Therefore, the critical parameter is

$$p_{\mathrm{lr}} = (\cos^2 2\alpha + 2^{N-2} \sin^2 2\alpha)^{-\frac{1}{2N}} \to \tfrac{1}{\sqrt{2}}. \tag{20}$$

The critical values of p decrease with the number of qubits, showing that many party generalized GHZ states are more and more robust against this type of noise. Finally, for $N \to \infty$ there is a Werner gap of $p \in (\frac{1}{2}, \frac{1}{\sqrt{2}})$ for which the entangled state does not improve the related communication complexity tasks.

3. Dephasing

Similar results hold for other local noises. After dephasing in the local z bases, the correlation tensor of the generalized

GHZ state reads

$$T_{\underbrace{y\ldots y}_{2k}x\ldots x} = (-1)^k \lambda^{\frac{N}{2}} \sin 2\alpha, \quad k = 0, 1, \ldots, \lfloor \tfrac{N-1}{2} \rfloor,$$
$$T_{\underbrace{z\ldots z}_{k}0\ldots 0} = \begin{cases} 1 & \text{for} \quad k \text{ even}, \\ \cos 2\alpha & \text{for} \quad k \text{ odd}. \end{cases} \tag{21}$$

All entangled generalized GHZ states are still entangled after the local dephasing. For a proof, it is sufficient to choose $G_{zz0\ldots 0} = G_{x\ldots xx} = 1$. For this choice, the right-hand side of (2) reads $R = 1 + \lambda^N \sin^2 2\alpha$, whereas for the left-hand side we have $L = \max(T_z^{(1)}T_z^{(2)} + T_x^{(1)}T_x^{(2)}\lambda^{N/2}\sin 2\alpha) \leqslant T_z^{(1)}T_z^{(2)} + T_x^{(1)}T_x^{(2)} \leqslant 1$, which follows from $\lambda^{N/2}\sin 2\alpha \leqslant 1$ and writing the components of the local tensors in polar coordinates. We also assumed that the local Bloch vectors of all the parties except first and second are along the x axis, which is optimal. Therefore, the state is entangled for all $\alpha > 0$ and $\lambda > 0$, independent of the number of qubits.

Since dephasing leaves the correlations in specific directions unchanged, violation of the Bell inequality for the generalized GHZ state is very robust against this type of noise. We show that it actually is state independent, i.e., violation is observed for all $\alpha > 0$ and only depends on the degree of dephasing if N is odd. Consider the multisetting condition, in which, as before, the last index takes values $\{y, z\}$ and the remaining indices are either $\{x, y\}$ if the last index is y, or z if the last index is z. If N is even, after dephasing, the state still contains perfect correlations in the z directions and some other correlations in the xy plane, and violates the inequalities for all $\alpha > 0$ and $\lambda > 0$. For the case of an odd number of qubits, the condition reads $\mathcal{D} = \cos^2 2\alpha + 2^{N-2}\lambda^N \sin^2 2\alpha$, and the violation is observed as soon as

$$\sin^2\alpha > 0 \quad \text{and} \quad \lambda > 2^{\frac{2}{N}-1} \to \tfrac{1}{2}. \tag{22}$$

Therefore, violation only depends on the degree of dephasing in the case of odd N, and again there is a finite Werner gap of $\lambda \in (0, \frac{1}{2})$ in the limit $N \to \infty$.

4. Amplitude damping

Finally, we consider independent local amplitude-damping channels. The elements of the decohered generalized GHZ states are the following:

$$T_{\underbrace{y\ldots y}_{2k}x\ldots x} = (-1)^k \gamma^{\frac{N}{2}} \sin 2\alpha, \quad k = 0, 1, \ldots, \lfloor \tfrac{N-1}{2} \rfloor$$
$$T_{\underbrace{z\ldots z}_{k}0\ldots 0} = \begin{cases} \cos^2\alpha + \bar{\gamma}^k \sin^2\alpha & \text{for} \quad k \text{ even}, \\ \cos^2\alpha - \bar{\gamma}^k \sin^2\alpha & \text{for} \quad k \text{ odd}, \end{cases} \tag{23}$$

where $\bar{\gamma} = 2\gamma - 1$. To prove the Werner gap, consider the metric with nonvanishing elements $G_{j_1\ldots j_N} = 1$ with $j_n = 1, 2$. The right-hand side of the entanglement criterion reads $R = 2^{N-1}\gamma^N \sin^2 2\alpha$. The left-hand side is $L = \gamma^{N/2} \sin 2\alpha \max[T_x^{(1)} \ldots T_x^{(N)} - T_y^{(1)}T_y^{(2)}T_x^{(3)} \ldots T_x^{(N)} - \ldots]$, with the maximum taken over local tensors with components from one plane. We write the elements of individual tensors in polar coordinates, i.e., $T_x^{(n)} = \cos\theta_n$ and $T_y^{(n)} = \sin\theta_n$, and recognize that expression in the bracket is now given by $\cos(\theta_1 + \cdots + \theta_N)$. Therefore, $L = \gamma^{N/2}\sin 2\alpha$, which translates into the critical parameter for entanglement

$$\gamma_{\text{ent}} = \frac{1}{4}\left(\frac{2}{\sin 2\alpha}\right)^{\frac{2}{N}} \to \frac{1}{4}. \tag{24}$$

The violation of both many-setting and two-setting inequalities is the same for higher number of qubits. The two-setting condition reveals the critical parameter

$$\gamma_{\text{lr}} = \frac{1}{2}\left(\frac{2}{\sin 2\alpha}\right)^{\frac{1}{N}} \to \frac{1}{2}. \tag{25}$$

For large N, the states for practically all $\alpha > 0$ violate the inequalities if $\Upsilon_{\text{lr}} > 1/2$ and there is a finite Werner gap of $\gamma \in (\frac{1}{4}, \frac{1}{2})$ in the limit $N \to \infty$.

D. *W* state

In this section, we study the W state, and we emphasize the properties distinguishing it from the class of generalized GHZ states. The W state involves a single excitation delocalized over all the qubits:

$$|W\rangle = \frac{1}{\sqrt{N}}(|10\ldots 0\rangle + |01\ldots 0\rangle + \cdots + |00\ldots 1\rangle). \tag{26}$$

It is permutationally invariant, i.e., any permutation of particles leaves the state unchanged. Therefore, to describe its correlation tensor it is sufficient to present just three elements. All other nonvanishing elements have indices being permutations of the indices of the following ones:

$$T_{\underbrace{z\ldots z}_{k}0\ldots 0} = 1 - \frac{2k}{N},$$
$$T_{yyz\ldots z0\ldots 0} = T_{xxz\ldots z0\ldots 0} = \frac{2}{N}. \tag{27}$$

1. White noise

Consider a W state mixed with white noise:

$$\rho = \Upsilon|W\rangle\langle W| + (1-\Upsilon)\frac{1}{2^N}\mathbb{1}. \tag{28}$$

Contrary to the case of the mixed GHZ state, this mixed state gives rise to a Werner gap in the limit of $N \to \infty$.

To prove entanglement of this state, consider the metric with nonvanishing elements $G_{j_1\ldots j_N} = 1$, where $j_n = \{x, z\}$. With this choice, the right-hand side of condition (2) reads $R = \Upsilon^2[1 + \binom{N}{2}\frac{4}{N^2}] = \Upsilon^2(3 - \frac{2}{N})$. The left-hand side is maximized if all the local tensors are along the $\pm z$ axis and equals $L = \Upsilon$, which we have verified numerically. Therefore, the critical parameter for entanglement reads

$$\Upsilon_{\text{ent}} = \frac{1}{3 - \frac{2}{N}} \to \frac{1}{3}. \tag{29}$$

The multisetting inequalities are violated as soon as the entanglement admixture is above the critical value [35]:

$$\Upsilon_{\text{lr}} = \sqrt{\frac{1}{3-\frac{2}{N}}} \to \frac{1}{\sqrt{3}}. \tag{30}$$

Note that the same correlations enter both the Bell inequalities and the entanglement criterion, showing that this simple application of the criterion is at least quadratically better in revealing entanglement than the Bell inequalities, i.e., $\Upsilon_{\text{ent}} \leqslant \Upsilon_{\text{lr}}^2$. This is a general feature present in all our examples.

2. Colored noise

Similar conclusion for comparison with the GHZ states follows in the case of colored-noise admixture to the W state:

$$\rho = \Upsilon|W\rangle\langle W| + (1-\Upsilon)|0\ldots 0\rangle\langle 0\ldots 0|, \tag{31}$$

where the colored noise introduces correlations in the local z directions.

This state is entangled for all $\Upsilon > 0$. A simple way to see this is to use the fact that if a subsystem is entangled, then the whole system is also entangled. The definition of the W state leads to the following form of the reduced density operator for any two qubits:

$$\Upsilon'|\psi^+\rangle\langle\psi^+| + (1-\Upsilon')|00\rangle\langle 00|, \tag{32}$$

with $\Upsilon' = \frac{2}{N}\Upsilon$ and $|\psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$. This state is entangled (has negative partial transposition [46,47]) for all $\Upsilon' > 0$. Therefore the global state is entangled for all $\Upsilon_{\text{ent}} > 0$ and any finite N.

If one chooses settings from the xz plane in the multisetting Bell inequality, the violation conditions reveal the critical value of the admixture parameter as given by

$$\Upsilon_{\text{lr}} = \frac{2}{3-\frac{1}{N}} \to \frac{2}{3}. \tag{33}$$

In the limit of $N \to \infty$ we find the Werner gap of $\Upsilon \in (0, \frac{2}{3})$. This is the biggest gap among all the states studied here.

3. Local depolarization

We shall show that the W state is very fragile with respect to this type of decoherence, in contrast to the GHZ state. After local depolarization, the elements of the correlation tensor of the decohered W state read

$$\begin{aligned} T_{\underbrace{z\ldots z}_{k}0\ldots 0} &= p^k\left(1-\frac{2k}{N}\right), \\ T_{yy\underbrace{z\ldots z}_{k}0\ldots 0} = T_{xx\underbrace{z\ldots z}_{k}0\ldots 0} &= p^{k+2}\frac{2}{N}. \end{aligned} \tag{34}$$

To prove entanglement of this state, consider nonzero metric elements $G_{j_1\ldots j_N} = 1/p^N$ with $j_n = \{x,z\}$. For this choice, the right-hand side of the criterion equals $R = p^N[1 + \binom{N}{2}\frac{4}{N^2}]$, whereas the maximum of the left-hand side is 1, which we have verified numerically. Therefore, the state is entangled above the critical value of

$$p_{\text{ent}} = \left(\frac{1}{3-\frac{2}{N}}\right)^{\frac{1}{N}} \to 1. \tag{35}$$

The multisetting Bell inequalities are violated as soon as $\mathcal{D} = T_{z\ldots z}^2 + \binom{N}{2}T_{xxz\ldots z}^2 > 1$. This gives the critical parameter

$$p_{\text{lr}} = \left(\frac{1}{3-\frac{2}{N}}\right)^{\frac{1}{2N}} \to 1, \tag{36}$$

which rapidly increases with N and already for five qubits requires $p > 0.9$. Since $p_{\text{ent}} = p_{\text{lr}}^2$, there is a Werner gap for all finite N, and in the limit both parameters tend to the same value. Of course, a smarter choice of the metric in the entanglement condition could prove that even in the limit there is a finite Werner gap.

4. Dephasing

The W state is extremely robust against dephasing, as it leaves the perfect correlations unchanged. After dephasing, the W state is transformed to

$$\begin{aligned} T_{\underbrace{z\ldots z}_{k}0\ldots 0} &= 1-\frac{2k}{N}, \\ T_{yy\underbrace{z\ldots z}_{k}0\ldots 0} = T_{xx\underbrace{z\ldots z}_{k}0\ldots 0} &= \lambda\frac{2}{N}. \end{aligned} \tag{37}$$

The dephased state violates Bell inequality (and therefore is entangled) for all N and all nontrivial dephasing channels. Consider correlations in the xz plane and multisetting condition. The value of parameter $\mathcal{D} = 1 + 2\lambda^2(1-\frac{1}{N})$ exceeds unity for all $\lambda > 0$. Note that this is true also in the limit $N \to \infty$.

5. Amplitude damping

The W state after this type of decoherence reads

$$\begin{aligned} T_{\underbrace{z\ldots z}_{k}0\ldots 0} &= 1-\frac{2k}{N}\gamma, \\ T_{yy\underbrace{z\ldots z}_{k}0\ldots 0} = T_{xx\underbrace{z\ldots z}_{k}0\ldots 0} &= \frac{2}{N}\gamma. \end{aligned} \tag{38}$$

To prove the Werner gap, consider nonvanishing metric elements $G_{j_1\ldots j_N} = 1$ for $j_n = \{x,z\}$. The right-hand side of the criterion is $R = (1-2\gamma)^2 + \binom{N}{2}\frac{4}{N^2}$, and the maximum of the left-hand side $L \leqslant 1-2\gamma$ is attained for all local vectors along z directions. Therefore, the state is entangled above the critical value

$$\gamma_{\text{ent}} = \frac{1}{3-\frac{1}{N}} \to \frac{1}{3}. \tag{39}$$

We check violation of Bell inequalities using the condition with many settings in the xz plane. The expression reads $\mathcal{D} = R$ and exceeds unity for all values above

$$\gamma_{\text{lr}} = \frac{2}{3-\frac{1}{N}} \to \frac{2}{3}. \tag{40}$$

The Werner gap is present also in the limit $N \to \infty$, just as for the GHZ state.

V. SUMMARY

Using the entanglement criterion [29], we have found families of entangled states which satisfy specific classes of Bell inequalities. We summarize our findings in Table II. Generally speaking, a simple application of the entanglement criterion gives at least quadratically better critical parameters than the

TABLE II. Summary of the results. The results for different initial states are presented in rows. Noisy channels applied to them are presented in columns. The strength of the noises is characterized by a single parameter ζ ($\zeta = 1$ corresponds to no noise, $\zeta = 0$ describes the strongest noise which immediately destroys entanglement). To unify presentation in this table, we represent all the parameters with ζ. In the main text, the parameter characterizing local depolarization is denoted by p, dephasing by λ, amplitude damping by γ, and admixture of white or colored noise by Υ. We present the critical parameter ζ_{ent}, above which the resulting state is entangled, and ζ_{lr}, below which the state satisfies classes of Bell inequalities, in the limit of large number of qubits $N \to \infty$. Additionally, to compare on an equal footing the critical parameters for different types of noises, they are all calculated here per particle in a sense that the values related to white and colored noise are Nth roots of the values of the main text. In all cases, ζ_{ent} is at most a square of ζ_{lr}.

	White noise	Colored noise	Local depolarization	Dephasing	Amplitude damping
Gen. GHZ	$\zeta_{\mathrm{ent}} \to 1/2$	$\zeta_{\mathrm{ent}} \to 0$	$\zeta_{\mathrm{ent}} \to 1/2$	$\zeta_{\mathrm{ent}} \to 0$	$\zeta_{\mathrm{ent}} \to 1/4$
	$\zeta_{\mathrm{lr}} \to 1/\sqrt{2}$	$\zeta_{\mathrm{lr}} \to 1/\sqrt{2}$	$\zeta_{\mathrm{lr}} \to 1/\sqrt{2}$	$\zeta_{\mathrm{lr}} \to 1/2$	$\zeta_{\mathrm{lr}} \to 1/2$
W	$\zeta_{\mathrm{ent}} \to 1$	$\zeta_{\mathrm{ent}} \to 0$	$\zeta_{\mathrm{ent}} \to 1$	$\zeta_{\mathrm{ent}} \to 0$	$\zeta_{\mathrm{ent}} \to 1/3$
	$\zeta_{\mathrm{lr}} \to 1$	$\zeta_{\mathrm{lr}} \to 1$	$\zeta_{\mathrm{lr}} \to 1$	$\zeta_{\mathrm{lr}} \to 0$	$\zeta_{\mathrm{lr}} \to 2/3$

ones obtained using the Bell inequalities. Therefore, we found entangled states satisfying the Bell inequalities in all studied cases. Moreover, we gave examples in which even in the limit of a large number of qubits, there is a finite gap between the critical parameter for entanglement and the critical parameter for Bell violation. We found that maximally entangled mixed states of two qubits give rise to the highest discrepancy between the critical parameters. It would be interesting to investigate if this also holds for a higher number of qubits. Our results are a further step toward full classification of entangled states into those which do and do not admit local realistic explanation.

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