Zero-error communication via quantum channels, non-commutative graphs and a quantum Lovász ϑ function

Runyao Duan,[1,2] Simone Severini,[3] and Andreas Winter[4,5]

[1]Centre for Quantum Computation and Intelligent Systems (QCIS), Faculty of Engineering and Information Technology, University of Technology, Sydney NSW2007, Australia

[2]State Key Laboratory of Intelligent Technology and Systems, Tsinghua National Laboratory for Information Science and Technology, Department of Computer Science and Technology, Tsinghua University, Beijing 100084, China

[3]Department of Physics and Astronomy, University College London, WC1E 6BT London, U.K.

[4]Department of Mathematics, University of Bristol, Bristol BS8 1TW, U.K.

[5]Centre for Quantum Technologies, National University of Singapore, 2 Science Drive 3, Singapore 117542*

(Dated: 11 March 2010)

We study the quantum channel version of Shannon's zero-error capacity problem. Motivated by recent progress on this question, we propose to consider a certain operator space as the quantum generalisation of the adjacency matrix, in terms of which the plain, quantum and entanglement-assisted capacity can be formulated, and for which we show some new basic properties.

Most importantly, we define a quantum version of Lovász' famous ϑ function, as the norm-completion (or stabilisation) of a "naive" generalisation of ϑ. We go on to show that this function upper bounds the number of entanglement-assisted zero-error messages, that it is given by a semidefinite programme, whose dual we write down explicitly, and that it is multiplicative with respect to the natural (strong) graph product.

We explore various other properties of the new quantity, which reduces to Lovász' original ϑ in the classical case, give several applications, and propose to study the operator spaces associated to channels as "non-commutative graphs", using the language of Hilbert modules.

I. CLASSICAL CHANNELS, GRAPHS AND ZERO-ERROR COMMUNICATION

For a classical channel $N : X \to Y$ between discrete alphabets X and Y (in the following assumed to be finite), i.e. a probability transition function $N(y|x)$, Shannon [31] initiated the study of zero-error capacities, i.e. of transmitting messages by one and asymptotically many uses of the channel.

To transmit messages through this channel with no probability of confusion, different messages m need to be associated to different input symbols x in such a way that the output distributions $N(\cdot|x)$ have disjoint supports. This motivates the introduction of the *confusability graph* G of N, that has the vertex set X and an edge $x \sim x'$ whenever x and x' can be confused via the channel, i.e. if there exists $y \in Y$ such that $N(y|x)N(y|x') \neq 0$. Clearly then, a code has to consist of an independent set (also known as stable set, or anti-clique) $X_0 \subset X$, i.e. a set of vertices without edges between them. The maximum size $|X_0|$ of an independent set in G is called the independence number $\alpha(G)$, and by the preceding discussion it is the maximum number of messages that can be transmitted through the channel without the possibility of confusing them.

Using two channels N_1 and N_2 in parallel means really that we have a product channel

$$N_1 \times N_2 : X_1 \times X_2 \to Y_1 \times Y_2, \quad \text{with} \quad (N_1 \times N_2)(y_1y_2|x_1x_2) = N_1(y_1|x_1)N_2(y_2|x_2).$$

*Electronic address: a.j.winter@bris.ac.uk

If the channels have confusability graphs G_1 and G_2, respectively, the confusability graph of the product channel is the (strong) graph product $G_1 \times G_2$ which has vertex set $X_1 \times X_2$ and edges

$$x_1x_2 \sim x_1'x_2' \quad \text{iff} \quad \begin{cases} \text{either } x_1 \sim x_1' \text{ and } x_2 \sim x_2', \\ \text{or } x_1 \sim x_1' \text{ and } x_2 = x_2', \\ \text{or } x_1 = x_1' \text{ and } x_2 \sim x_2'. \end{cases}$$

(If this looks complicated, it does so because it has to encapsulate the idea that a symbol can be confused with itself.) An integer n uses a channel with confusability graph G is thus described by the n-fold graph product G^n. With this we can define the zero-error capacity of the graph as

$$C_0(G) = \lim_{n\to\infty} \frac{1}{n} \log \alpha(G^n) = \sup_n \frac{1}{n} \log \alpha(G^n),$$

i.e. the asymptotically largest number of bits transmissible with certainty, per channel use (throughout, $\log$ is understood as the binary logarithm). Note that in graph theory the convention is prevalent to call $\Theta(G) := 2^{C_0(G)} = \sup_n \sqrt[n]{\alpha(G^n)}$ the zero-error capacity, but in this paper we prefer to stay in keeping with the information theoretic usage.

For some graphs, $C_0(G) = \log \alpha(G)$, but in general the zero-error capacity is larger – a well-known example is the pentagon C_5 whose capacity is $\frac{1}{2}\log 5$ [26], and there are graphs such that for every finite n, $\frac{1}{n}\log \alpha(G^n) < C_0(G)$ [19]. Finding $\alpha(G)$ (and a maximal-size independent set) is in general an NP-hard problem, and the calculation of the zero-error capacity is even worse as it is not even known whether $C_0(G)$ is computable.

It should be noted that Shannon [31] also considered (and solved) the problem of zero-error transmission via many realisations of N in the presence of instantaneous (passive and noiseless) feedback. In that case, it is not sufficient to look at the confusability graph G of the channel, but rather at the full bipartite graph that represents the possible input-output transitions. The capacity $C_{0\mathrm{F}}(N)$ in that case is either 0, if $C_0(N) = 0$, or given by the logarithm of a linear programming relaxation of the independence number, called *fractional packing number*. Note that $C_{0NS}(N)$, the zero-error capacity in the presence of arbitrary non-signalling correlations [3] has the same property, and in fact is always the logarithm of the fractional packing number [12].

A much better upper bound on $\alpha(G)$ was given by Lovász [26] as a semidefinite programming relaxation, and called $\vartheta(G)$: rephrasing slightly [26, Thms. 5 and 6],

$$\vartheta(G) = \max\big\{ \|\mathbb{1} + T\| : T_{xx'} = 0 \text{ if } x = x \text{ or } x \sim x', \text{ and } \mathbb{1} + T \geq 0 \big\}, \tag{1}$$

where the maximum is over $|X| \times |X|$ complex (Hermitian) matrices T, though one can show that it is sufficient to consider *real symmetric* A in the above formula. In fact, via an expression of ϑ as the solution to a semidefinite programme, it can also be shown to be multiplicative with respect to the graph product (i.e. $\vartheta(G \times H) = \vartheta(G)\vartheta(H)$). Thus, it also gives an upper bound $C_0(G) \leq \log \vartheta(G)$ on the zero-error capacity. Apart from some special graphs exhibited by Haemers [20] and a particular construction by Alon [1], it remains the best upper bound on the zero-error capacity, and has been deeply studied ever since it appeared [23].

In the rest of the paper we will extend this theory to quantum channels and structures generalising the confusability graph (see section II). Instead of introducing only the mathematical objects, we shall precede each definition by a motivating discussion of the zero-error information theory; for instance in section III we will introduce zero-error codes for channels to motivate our definitions of quantum independence numbers (there are at least three meaningful ones). Then in section IV, we introduce the quantum ϑ function, explore some of its properties, of which the most important one is the semidefinite formulation (section V). We end with highlighting several applications (section VI), and discussing future directions with non-commutative graphs, in section VII, where we propose an algebraic framework for them.

II. QUANTUM CHANNELS AND NON-COMMUTATIVE GRAPHS

To describe the quantum generalisations of the above combinatorial concepts, we start with quantum communication channels, mapping quantum states to quantum states. The input and output alphabets of a channel are replaced by (complex) Hilbert spaces A and B – in the present paper of finite dimension $|A|$ and $|B|$ – with their spaces of linear operators $\mathcal{L}(A)$, etc. The Hermitian (self-adjoint) operators $\mathcal{L}(A)_{\text{sa}}$ are the physical observables on A, while the states are the *density operators* $\rho \in \mathcal{S}(A) \subset \mathcal{L}(A)$, i.e. $\rho \geq 0$ and $\operatorname{Tr}\rho = 1$. Note that the set of states is a convex body whose extreme points are exactly the one-dimensional projectors $|\psi\rangle\langle\psi|$ onto one-dimensional subspaces $\mathbb{C}|\psi\rangle$ with a unit vector $|\psi\rangle \in A$. [We use Dirac notation throughout: $|\psi\rangle \in A$ is a vector, $\langle\psi|$ it's adjoint (a linear form), $\langle\varphi|\psi\rangle$ denotes the Hilbert space inner product (linear in the second argument), and $|\psi\rangle\langle\varphi|$ is the corresponding outer product, a rank one operator in $\mathcal{L}(A)$, etc.] A quantum channel is now a linear map $\mathcal{N} : \mathcal{L}(A) \to \mathcal{L}(B)$ that is additionally *completely positive and trace preserving (cptp)*. The latter means that $\operatorname{Tr}\mathcal{N}(\rho) = \operatorname{Tr}\rho$; the former means that not only $\mathcal{N}$ maps positive semidefinite operators into positve semidefinite operators (being a "positive" map, for short), but also all extensions $\mathcal{N} \otimes \mathrm{id}_R$ for an arbitrary Hilbert space R. The class of completely positive maps is the largest subset of positive maps containing the identity and stable under tensor products [17].

Cptp maps between Hilbert space operator algebras have several useful representations with associated physical interpretation. One of them is the *Kraus form* $\mathcal{N}(\rho) = \sum_j E_j \rho E_j^\dagger$ with Kraus operators $E_j : A \to B$, which can be read as the state change under a generalised measurement with "events" j. Every such form defines a completely positive map, and it is trace preserving iff $\sum_j E_j^\dagger E_j = \mathbb{1}$.

Classical channels are embedded into this picture as follows: starting from the sample space, e.g. the inputs X to a channel, we consider the Hilbert space $\mathbb{C}X$, spanned by the orthonormal basis $\{|x\rangle\}_{x\in X}$. The probability simplex is mapped to the convex hull of the pure basis states $|x\rangle\langle x|$, so that we focus only on density operators diagonal in the computational basis. A classical channel $N : X \to Y$ has to be translated into a cptp map between the diagonal matrices over $A = \mathbb{C}X$ and $B = \mathbb{C}Y$, which is done canonically by constructing it from the Kraus operators $\sqrt{N(y|x)}|y\rangle\langle x|$, $x \in X$, $y \in Y$. I.e., for each classical probabilistic transition $x \to y$ there is an event in the quantum cptp map.

For the channel $\mathcal{N} : \mathcal{L}(A) \to \mathcal{L}(B)$, with Kraus operators $E_j : A \to B$, we now define the *non-commutative (confusability) graph* as the operator subspace

$$S := \operatorname{span}\{E_j^\dagger E_k : j, k\} < \mathcal{L}(A). \tag{2}$$

In [11, 15] it is shown that a subspace S is associated in the above way to a channel iff $\mathbb{1} \in S$ and $S = S^\dagger$. That is why we shall call an operator space $S < \mathcal{L}(A)$ with these properties a *non-commutative graph*, regarding the operator space S as the quantum generalization of the classical confusability graph G. This idea is enforced by the observation that for two channels $\mathcal{N}_1$ and $\mathcal{N}_2$, with associated subspaces S_1 and S_2, respectively, the tensor product channel $\mathcal{N}_1 \otimes \mathcal{N}_2$ has operator subspace $S_1 \otimes S_2$. We shall come back to this notion, with a proper (and more subtle) definition, in the last section VII.

Again, let us review this concept in the classical case: as we have seen, the Kraus operators may be chosen as $E_{xy} = \sqrt{N(y|x)}|y\rangle\langle x|$, meaning that

$$E_{x'y'}^\dagger E_{xy} = \sqrt{N(y'|x')N(y|x)}\langle y'|y\rangle|x'\rangle\langle x|$$

is nonzero iff $y = y'$ and $N(y|x')N(y|x) \neq 0$. Thus,

$$S = \{T : \forall x \not\sim x' \; \langle x|T|x'\rangle = 0\},$$

which means that from the patterns of zeros in the $|X| \times |X|$-matrix representation of the admissible T we can read off the graph complement $\overline{G}$ of the confusability graph G. Note that an operator space such as this is always a non-commutative graph, and that there is always a classical channel N giving rise to S: simply choose as the output alphabet Y the set of edges of G, and N maps an input symbol to a random edge incident with it.

Coming back to the general case: An alternative way of thinking about the state change due to the channel $\mathcal{N}$ is to view it as a pulling-back of observables on B to observables on A: the linear map effecting this translation is the adjoint $\mathcal{N}^*(X) = \sum_j E_j^\dagger X E_j$ (in physics often called the "Heisenberg picture", in contrast to the "Schrödinger picture" $\mathcal{N}$), and indeed one may think of the channel $\mathcal{N}$ as allowing the receiver to make (generally distorted) measurements on A. The adjoint map is characterised by being completely positive and *unital*, i.e. $\mathcal{N}^*(\mathbb{1}) = \mathbb{1}$.

Every channel has a Stinespring dilation, representing the dynamics as an isometry followed by a partial trace: i.e., there exists $V : A \hookrightarrow B \otimes C$ such that

$$\mathcal{N}(\rho) = \mathrm{Tr}_C\, V\rho V^\dagger,$$

and up to isometric equivalence, C (the "environment") and V are unique. Then one has a unique *complementary channel*

$$\widehat{\mathcal{N}}(\rho) = \mathrm{Tr}_B\, V\rho V^\dagger,$$

representing the information loss of the original channel to the environment. Note that the adjoint maps of $\mathcal{N}$ and $\widehat{\mathcal{N}}$ can be written compactly using the Stinespring isometry V:

$$\mathcal{N}^*(X) = V^\dagger(X \otimes \mathbb{1})V,$$
$$\widehat{\mathcal{N}}^*(Y) = V^\dagger(\mathbb{1} \otimes Y)V.$$

recalling that $V^\dagger$ is a projection.

Lemma 1 *For any channel $\mathcal{N}$ with complementary channel $\widehat{\mathcal{N}}$, $S = \widehat{\mathcal{N}}^*\big(\mathcal{L}(C)\big)$. In words: S is the space of operators on A measurable by the channel environment.*

Proof We can write a Stinespring dilation of $\mathcal{N}$ via the injective $V : A \hookrightarrow B \otimes C$,

$$V|\varphi\rangle = \sum_j (E_j|\varphi\rangle)^B \otimes |j\rangle^C,$$

so that for an arbitrary operator $X \in \mathcal{L}(C)$ the Heisenberg map of the complementary channel reads

$$\widehat{\mathcal{N}}^*(X) = V^\dagger(\mathbb{1} \otimes X)V = \sum_{j,k} E_j^\dagger E_k \langle j|X|k\rangle.$$

Now, since the operators $|k\rangle\langle j|$ form a basis of $\mathcal{L}(C)$,

$$\widehat{\mathcal{N}}^*\big(\mathcal{L}(C)\big) = \mathrm{span}\big\{E_j^\dagger E_k : j,k\big\} = S,$$

i.e., the image of $\widehat{\mathcal{N}}^*$ is indeed S. □

Note that this lemma also shows that our definition of S was sound: it doesn't depend on the particular choice of Kraus operators, and can be entirely understood in terms of the channel map (or rather its complement).

Remark In general, S does not uniquely define the channel $\mathcal{N}$ from which it originates. Already classical graphs and channels show this, as the confusability graphs records only which pairs of inputs can lead to the same output with the same probability, but it doesn't remember the value of this probability, nor can it tell us about the triples of inputs which can end up at the same output (note that even if there is a triangle in G, there may not be a single output symbol which can be reached by all of its vertices).

Returning to the channel motivation, we can ask what happens to a non-commutative graph $S = \widehat{\mathcal{N}}^*(\mathcal{L}(C))$ if we add post-processing or pre-processing to the channel $\mathcal{N} : \mathcal{L}(A) \rightarrow \mathcal{L}(B)$. I.e., considering channels $\mathcal{R} : \mathcal{L}(B) \rightarrow \mathcal{L}(B')$ and $\mathcal{T} : \mathcal{L}(A') \rightarrow \mathcal{L}(A)$, let us look at the non-commutative graphs $\widehat{S} < \mathcal{L}(A)$ and $S' < \mathcal{L}(A')$ belonging to the compositions $\mathcal{R}\circ\mathcal{N}$ and $\mathcal{N}\circ\mathcal{T}$, respectively.

Regarding the former, looking at the definition eq. (2) shows that $S < \widehat{S}$, which is the natural relation of S being a subgraph of $\widehat{S}$. Regarding the latter, fix a Stinespring isometry $U : A' \hookrightarrow A \otimes D$, and observe that

$$S' = \widehat{\mathcal{N}\circ\mathcal{T}}^* \big(\mathcal{L}(C \otimes D)\big) = U^\dagger \big(S \otimes \mathcal{L}(D)\big) U.$$

The projection $U^\dagger : A \otimes D \rightarrow A'$ can be understood as giving rise to an *induced* subgraph (much as a subset of the vertices of a classical graph would). Note that in this way, every non-commutative graph S is an induced subgraph of the product $\mathbb{1}_B \otimes \mathcal{L}(C)$ of an empty and a complete graph, by virtue of the Stinespring dilation V of an appropriate channel $\mathcal{N}$. We come back to the issue of (induced) subgraphs again in section VII.

III. ZERO-ERROR COMMUNICATION WITH AND WITHOUT ENTANGLEMENT

Zero-error information transmission via general quantum channels was considered first by Medeiros *et al.* [27], and then by Beigi and Shor [5] (in those investigations, communication signals were, implicitly or explicitly, restricted to product states across multiple channel uses); more recently in full generality by Cubitt, Chen and Harrow [11], Duan [15] and Cubitt and Smith [13]; Duan and Shi [16] present results on multi-user quantum zero-error capacity, while quantum effects for classical channels were discovered by Cubitt, Leung, Matthews and Winter [12].

Let $\mathcal{N} : \mathcal{L}(A) \rightarrow \mathcal{L}(B)$ be a quantum channel, i.e. a linear c.p.t.p. map, with Kraus operators $E_j : A \rightarrow B$, so that $\mathcal{N}(\rho) = \sum_j E_j \rho E_j^\dagger$. Then to send messages m one has to associate them with states ρ such that different states ρ, σ lead to orthogonal channel output states: $\mathcal{N}(\rho) \perp \mathcal{N}(\sigma)$, because it is precisely the orthogonal states that can be distinguished with certainty. Clearly, these states may, w.l.o.g., be taken as pure, as the orthogonality is preserved when going to any states in the support (i.e., the range) of ρ, σ, etc.

Now we make the elementary observation, made in previous work, that two input pure states $\varphi = |\varphi\rangle\langle\varphi|$ and $\psi = |\psi\rangle\langle\psi|$ for unit vectors $|\varphi\rangle, |\psi\rangle \in A$, lead to orthogonal output states $\mathcal{N}(\varphi)$ and $\mathcal{N}(\psi)$ iff

$$0 = \operatorname{Tr} \mathcal{N}(\varphi)\mathcal{N}(\psi) = \sum_{jk} \big|\langle\varphi|E_j^\dagger E_k|\psi\rangle\big|^2,$$

which says that for all $j,\ k$, $\langle\varphi|E_j^\dagger E_k|\psi\rangle = 0$. In other words,

$$|\varphi\rangle\langle\psi| \perp S = \operatorname{span}\big\{E_j^\dagger E_k : j, k\big\},$$

the non-commutative confusability graph of the channel $\mathcal{N}$, where the orthogonality is with respect to the Hilbert-Schmidt inner product $\operatorname{Tr} A^\dagger B$ of operators.

From the above formula it is clear that the maximum number $\alpha(\mathcal{N})$ of one-shot zero-error distinguishable messages down the channel is given as the maximum size of a set of (orthogonal) vectors $\{|\phi_m\rangle : m = 1, \dots, N\}$ such that

$$\forall m \neq m' \quad |\phi_m\rangle\langle\phi_{m'}| \in S^\perp. \tag{3}$$

Since it is only a property of S, we shall denote $\alpha(\mathcal{N})$ also as $\alpha(S)$, and we call it the *independence number* of S. Note that the defining property of the operators $|\phi_m\rangle\langle\phi_{m'}|$ in eq. (3) is that they are rank-one and an orthonormal system orthogonal to S, with respect to the Hilbert-Schmidt inner product. [In [5] it was proved that computing the independence number $\alpha(S)$ is QMA-complete, much like $\alpha(G)$ is known to be NP-complete for graphs.]

There are at least two further reasonable notions of independence number possible for quantum channels and their confusability graphs. They are motivated by entanglement-assisted zero-error communication, and by the zero-error transmission of quantum information.

First, to transmit quantum information, one needs a subspace A' of A with projection operator P such that $PSP = \mathbb{C}P$ – this is exactly the Knill-Laflamme error correction condition [22]. For the channel this is precisely the necessary and sufficient condition for the existence of a decoding cptp map $\mathcal{D} : \mathcal{L}(B) \rightarrow \mathcal{L}(A')$ such that the composition

$$\mathcal{L}(A') \hookrightarrow \mathcal{L}(A) \xrightarrow{\mathcal{N}} \mathcal{L}(B) \xrightarrow{\mathcal{D}} \mathcal{L}(A')$$

is the identity map. Let $\alpha_q(S)$ be the largest dimension of such a quantum error correcting code A', which we call the *quantum independence number*.

Finally, $\widetilde{\alpha}(S)$ is defined to be the largest integer N such that there exist Hilbert spaces A_0 and B_0, a state $\omega \in \mathcal{S}(A_0 \otimes B_0)$ and cptp maps $\mathcal{E}_m : \mathcal{L}(A_0) \rightarrow \mathcal{L}(A)$ $(m = 1, \dots, N)$ such that the N states $\rho_m = (\mathcal{N} \circ \mathcal{E}_m \otimes \mathrm{id}_{B_0})\omega$ are pairwise orthogonal. This definition of the *entanglement-assisted independence number* is motivated by the scenario where sender and receiver share the state ω beforehand, and the sender uses the encoding maps $\mathcal{E}_m$ to modulate the state before sending her share into the channel. The receiver has to be able to recover the message from his final state, ρ_m. As before, we can argue that the shared state is w.l.o.g. pure, i.e. $\omega = |\Omega\rangle\langle\Omega|$ for a unit vector $|\Omega\rangle \in A_0 \otimes B_0$, either by picking $|\Omega\rangle$ from the support of ω, or by purification. In this way, we can already assume $A_0 \simeq B_0$. Furthermore, all $\mathcal{E}_m$ have Stinespring dilations $V_m : A_0 \hookrightarrow A \otimes R$ (w.l.o.g. using the same extension R), so that $\mathcal{E}_m(\rho) = \operatorname{Tr}_R V\rho V^\dagger$. Now it is easily seen that the orthogonality condition on the ρ_m is equivalent to the the states

$$|\phi_m\rangle = (V_m \otimes \mathbb{1})|\Omega\rangle \in A \otimes R \otimes B_0$$

forming an independent set for $S \otimes \mathcal{L}(R) \otimes \mathbb{1}_{B_0}$. We can reformulate this in turn without referring to B_0, by noting that there is a state $\rho \in \mathcal{S}(A \otimes R)$ and unitaries U_m on $A \otimes R$ such that

$$\operatorname{Tr}_{B_0} |\phi_m\rangle\langle\phi_{m'}| = U_m \rho U_{m'}^\dagger,$$

so that we may regard an entanglement-assisted independent set as a state $\rho \in \mathcal{S}(A \otimes R)$ and a collection of unitaries U_m, such that for all $m \neq m'$, $U_m \rho U_{m'}^\dagger \perp S \otimes \mathcal{L}(R)$.

A special case is when the encoding modulation is only unitary, and the extension system is trivial, $R = \mathbb{C}$. The largest number of messages under this additional restriction we denote $\widetilde{\alpha}_U(S)$, and call it the *unitary entanglement-assisted independence number*.

On the other hand, if we lift the restriction that the encoding maps $\mathcal{E}_m$ have to be trace preserving (but demanding it for the decoding), we obtain the *generalised entanglement-assisted independence number* $\widehat{\alpha}(S)$: we demand instead that $\mathcal{E}_m(\sigma) = \sum_j E_{jm} \sigma E_{jm}^\dagger$ is such that $\sum_j E_{jm}^\dagger E_{jm} \in$

$\mathrm{GL}(A_0)$ is invertible. Since such cp maps still have a Stinespring dilation, only that V is no longer isometry by invertible, we arrive at the notion of a *generalised entanglement-assisted independent set*, consisting as before of a state $\rho \in \mathcal{S}(A \otimes R)$ and invertible operators $W_m \in \mathrm{GL}(A \otimes R)$, such that for $m \neq m'$, $W_m \rho W_{m'}^\dagger \perp S \otimes \mathcal{L}(R)$. Note that this concept even makes sense for *generalised non-commutative graphs*: $S = S^\dagger$, without the condition that $\mathbb{1} \in S$ but only assuming that there is a positive definite element $M \in S$.

The following proposition records some elementary properties of the independence numbers.

Proposition 2 *For all non-commutative graphs $S < \mathcal{L}(A)$,*

$$\alpha_q(S) \leq \alpha(S) \leq \widetilde{\alpha}_U(S) \leq \widetilde{\alpha}(S) \leq \widehat{\alpha}(S).$$

If $\dim S^\perp < k(k-1)$, then $\alpha(S) < k$; in particular $\alpha(S) \leq |A|$. Furthermore, $\widehat{\alpha}(S) \leq 1 + \dim S^\perp$, even for generalised non-commutative graphs.
Finally, all these independence numbers are monotonic (non-increasing) under pre- and postprocessing (see section II).

Proof The ordering of the five numbers is clear from the definition, and so is the monotonicity.

For the bound on $\alpha(S)$, note that an independent set requires $|\phi_m\rangle\langle\phi_{m'}| \in S^\perp$, for $\mathbb{1} \leq m \neq m' \leq k$. But these operators are clearly mutually orthogonal with respect to the Hilbert-Schmidt inner product, hence $k(k-1) \leq \dim S^\perp$.

For the bound on $\widehat{\alpha}(S)$, we first present a simple argument for $\widetilde{\alpha}_U(S)$: Consider an entanglement-assisted independent set with trivial R: we need a state $\rho \in \mathcal{S}(A)$ and unitaries $U_1, \ldots, U_N$ such that for all $m \neq m'$, $U_m \rho U_{m'}^\dagger \in S^\perp$. In particular, all the operators $U_m \sqrt{\rho}$ are mutually orthogonal, implying their linear independence. But then also the $U_m \rho U_1^\dagger$ are linearly independent, and they are all in $S^\perp$. Thus, $N - 1 \leq \dim S^\perp$.

To bound the generalised independence number, assume that there are N cp maps $\mathcal{E}_1, \ldots, \mathcal{E}_N$ from $\mathcal{L}(A_0)$ to $\mathcal{L}(A)$, and a state $\rho \in \mathcal{S}(A_0)$. In order to write out the condition for a generalised independent set explicitly, let us make some additional assumptions. First, we can write $\mathcal{E}_m(\sigma) = \sum_{j=1}^{r(m)} E_{mj} \sigma E_{mj}^\dagger$, where $\{E_{mj} : A_0 \rightarrow A\}_{k=1,\ldots,r(m)}$ is a set of Kraus operators for $\mathcal{E}_m$. It is convenient to denote $\mathcal{K}_m := \mathrm{span}\{E_{mj} : 1 \leq j \leq r(m)\}$, the Kraus operator space of $\mathcal{E}_m$, and $d_m := \dim \mathcal{K}_m$. Furthermore, $E_m := \sum_j E_{mj}^\dagger E_{mj}$ is a positive definite operator in $\mathcal{L}(A_0)$. We can also assume that ρ is invertible in $\mathcal{L}(A_0)$ as we can always choose A_0 to be the support of ρ without changing N. Now by a simple calculation, the condition for a generalised independent set can be rewritten as follows:

$$E_{mj} \rho E_{m'k}^\dagger \in S^\perp, \quad \text{for } 1 \leq m \neq m' \leq N,\ 1 \leq j \leq r(m),\ 1 \leq k \leq r(m').$$

Noticing that there is a positive definite operator $M \in S$, this implies

$$\mathrm{Tr}(M E_{mj} \rho E_{m'k}^\dagger) = 0, \quad \text{for } 1 \leq m \neq m' \leq N,\ 1 \leq j \leq r(m),\ 1 \leq k \leq r(m').$$

That is, the spaces of linear operators $\mathcal{K}_m \sqrt{\rho} = \mathrm{span}\{E_{mj}\sqrt{\rho} : 1 \leq j \leq r(m)\}$ are mutually orthogonal for different m, with respect to the generalised inner product given by $\langle X, Y \rangle_M = \mathrm{Tr}(M X Y^\dagger)$. In particular, the $\mathcal{K}_m$ are linearly independent, and $\mathcal{K}_m \cap \mathcal{K}_{m'} = 0$ for $m \neq m'$. Now let us focus on $m' = 1$, and w.l.o.g. assume $d_1 \leq d_m$ for all m. Then we have

$$E_{mj} \rho E_{1k}^\dagger =: X_{mj,k} \in S^\perp, \quad \text{for } 2 \leq m \leq N,\ 1 \leq j \leq r(m),\ 1 \leq k \leq r(1).$$

Multiplying both sides of the above from the right by E_{1k}, and summing over k, we obtain

$$E_{mj}\rho E_1 = X_{mj} := \sum_{k=1}^{r(1)} X_{mj,k}E_{1k} \in S^\perp \mathcal{K}_1,$$

for $2 \leq m \leq N$ and $1 \leq j \leq r(m)$. Since ρ and E_1 are invertible, this can be rewritten as

$$E_{mj} \in S^\perp \mathcal{K}_1(\rho E_1)^{-1}, \quad \text{for } 2 \leq m \leq N,\ 1 \leq j \leq r(m),$$

or equivalently $\sum_{m=2}^N \mathcal{K}_m < \operatorname{span}\{S^\perp \mathcal{K}_1\}(\rho E_1)^{-1}$. Noticing that

$$\dim \operatorname{span}\{S^\perp \mathcal{K}_1\} \leq (\dim S^\perp)(\dim \mathcal{K}_1) = d_1 \dim(S^\perp),$$

and $\dim \sum_{m=2}^N \mathcal{K}_m = \sum_{m=2}^N d_m$ (because of linear independence of the $\mathcal{K}_m$), we finally arrive at $(N-1)d_1 \leq \sum_{m=2}^N d_m \leq d_1 \dim(S^\perp)$, completing the proof. □

About the independence numbers $\alpha(S)$, $\alpha_q(S)$ and $\widetilde{\alpha}(S)$, and their associated operational capacities

$$\begin{aligned}
C_0(S) &= \lim_{n\to\infty} \frac{1}{n} \log \alpha\big(S^{\otimes n}\big) && \text{[(classical) zero-error capacity]},\\
Q_0(S) &= \lim_{n\to\infty} \frac{1}{n} \log \alpha_q\big(S^{\otimes n}\big) && \text{[quantum zero-error capacity]},\\
C_{0E}(S) &= \lim_{n\to\infty} \frac{1}{n} \log \widetilde{\alpha}\big(S^{\otimes n}\big) && \text{[entanglement-assisted zero-error capacity]},
\end{aligned}$$

quite a bit is known: In [15] examples of S are found such that $\alpha(S) = 1$ but $\alpha(S \otimes S) \geq 2$, and examples of S_1 and S_2 such that $C_0(S_1) = 0$ but $C_0(S_1 \otimes S_2) \gg C_0(S_2)$; furthermore, non-commutative graphs S such that $C_0(S) = 0$ but $C_{0E}(S) \geq 1$ (all of which are impossible for classical graphs). In fact, while $\alpha(S)$ can be 1 (and even $C_0(S) = 0$) for highly nontrivial graphs S, any non-trivial $S \lneqq \mathcal{L}(A)$ (i.e. not a complete graph) is easily seen to have $\widetilde{\alpha}(S) \geq \widetilde{\alpha}_U(S) \geq 2$. In [11] even non-commutative graphs S_1 and S_2 are shown to exist such that $C_0(S_1) = C_0(S_2) = 0$, yet $\alpha(S_1 \otimes S_2) \geq 2$; this result is further improved in [13] to yield even $\alpha_q(S_1 \otimes S_2) \geq 2$.

We can similarly define the generalised entanglement-assisted zero-error capacity

$$\widehat{C}_{0E}(S) = \lim_{n\to\infty} \frac{1}{n} \log \widehat{\alpha}\big(S^{\otimes n}\big),$$

which is evidently an upper bound on $C_{0E}(S)$. Both $\widehat{\alpha}(S)$ and $\widehat{C}_{0E}(S)$ have, by their very definition, an important symmetry property: for any invertible $W \in \mathrm{GL}(A)$,

$$\widehat{\alpha}(S) = \widehat{\alpha}(WSW^\dagger) \quad \text{and} \quad \widehat{C}_{0E}(S) = \widehat{C}_{0E}(WSW^\dagger), \tag{4}$$

which are meaningful because $S' = WSW^\dagger$ contains $WW^\dagger > 0$, and $S' = S'^\dagger$. (Note at the same time that $\widetilde{\alpha}(S)$ and $C_{0E}(S)$ satisfy these equations only if W is a unitary.) We call S and S' as above *congruent*, $S \simeq S'$, and denote the congruence class of S by $[S]$.

The independence numbers $\alpha_q(S)$, $\alpha(S)$, $\widetilde{\alpha}(S)$ and $\widehat{\alpha}(S)$ are computable: this is obvious for the first two, since they are formulated in terms of the solvability of a set of real polynomial equations and inequalities in a finite number of variables. For the latter two, the potentially unbounded dimension of the entangled state needed appears to create an issue. However, noting that the existence of a zero-error code with N messages can be cast as real algebraic problem in *non-commuting* variables with polynomial constraints, we can invoke recent results by Pironio *et* *al.* [28]: these state that a certain hierarchy of semidefinite programmes asymptotically characterises the solvability of such constraints by Hilbert space operators for some sufficiently large dimension. More precisely, one finds zero-error codes by solving polynomial equations for increasingly higher dimensional entangled states and measurements, and finds increasingly better witnesses that certain numbers of messages cannot be sent with zero-error by climbing higher in the hierarchy.

The algorithms implicit in these remarks are very inefficient (in fact, we cannot even give an upper bound on the runtime for $\widetilde{\alpha}$ and $\widehat{\alpha}$), but apart from the QMA-completeness of α no results concerning the complexity of the independence numbers have been reported. In contrast, as far as we know, none of the asymptotic capacities are even known to be decidable – cf. [2].

IV. A QUANTUM LOVÁSZ FUNCTION

For the any non-commutative graph $S < \mathcal{L}(A)$, i.e. $\mathbb{1} \in S$ and $S = S^\dagger$, we make, motivated by eq. (1), the following definition.

$$\vartheta(S) := \max\bigl\{\|\mathbb{1} + T\| : T \in S^\perp,\ \mathbb{1} + T \geq 0\bigr\}, \tag{5}$$

where the norm is the operator norm (i.e. the largest singular value). Note that all elements in $S^\perp$ are traceless, hence for $d = |A|$ the norm on the right hand side is at most that of the case where T has $d-1$ eigenvalues -1 and a single eigenvalue $d-1$; hence $\vartheta(S) \leq |A|$.

By the discussion in sections I and II, for a classical channel $N : X \to Y$ with confusability graph G, we can model the channel as a cptp map with Kraus operators $\sqrt{N(y|x)}|y\rangle\langle x|$, so S is spanned by all $|x'\rangle\langle x|$ such that $x \sim x'$ or $x = x'$. Thus, the space $S^\perp$ is exactly the set of matrices T with zeros in all entries $T_{xx'} = 0$ whenever $xx' \in G$ or $x = x'$. Thus, the eligible $\mathbb{1} + T$ in the definition (5) are positive semidefinite matrices with ones along the diagonal and zeroes in all entries (x, x') where x and x' are confusable. The maximum norm in eq. (5) coincides thus with the expression for $\vartheta(G)$ in [26, Thms. 5 and 6], and we conclude that $\vartheta(S) = \vartheta(G)$.

The above definition has some desirable properties:

Lemma 3 *For any non-commutative graph S, $\alpha(S) \leq \vartheta(S)$. Furthermore, ϑ is monotonic with respect to subgraphs, i.e. when $S \subset S'$ for two non-commutative graphs, then $\vartheta(S) \geq \vartheta(S')$.*

Proof The monotonicity is clear from the definition. For the relation to α, consider a maximal size indendent set $\{|\phi_m\rangle : m = 1, \ldots, N\}$, i.e. $N = \alpha(S)$. Then, $T = \sum_{m \neq m'} |\phi_m\rangle\langle\phi_{m'}| \in S^\perp$. Furthermore,

$$\mathbb{1} + T \geq \sum_m |\phi_m\rangle\langle\phi_m| + \sum_{m \neq m'} |\phi_m\rangle\langle\phi_{m'}| = \sum_{m,m'} |\phi_m\rangle\langle\phi_{m'}| \geq 0,$$

so T is eligible in the definition of $\vartheta(S)$. On the other hand,

$$\|\mathbb{1} + T\| = \left\| \sum_{m,m'} |\phi_m\rangle\langle\phi_{m'}| \right\| = N,$$

and we are done. □

Lemma 4 *ϑ is supermultipicative, i.e. for non-commutative graphs $S_1 < \mathcal{L}(A_1)$ and $S_2 < \mathcal{L}(A_2)$,*

$$\vartheta(S_1 \otimes S_2) \geq \vartheta(S_1)\vartheta(S_2).$$

Proof Observing that the operator subspace associated to the tensor product $\mathcal{N}_1 \otimes \mathcal{N}_2$ of channel $\mathcal{N}_i$ with operator subspaces S_i, respectively, is given by $S_1 \otimes S_2$, we can show

Namely, for $T_i \in S_i^\perp$ and $\mathbb{1} + T_i \geq 0$, we have

$$T := T_1 \otimes \mathbb{1} + \mathbb{1} \otimes T_2 + T_1 \otimes T_2 \in S_1^\perp \otimes S_2 + S_1 \otimes S_2^\perp + S_1^\perp \otimes S_2^\perp = (S_1 \otimes S_2)^\perp,$$

and $\mathbb{1} + T = (\mathbb{1} + T_1) \otimes (\mathbb{1} + T_2) \geq 0$. At the same time it follows that $\|\mathbb{1} + T\| = \|\mathbb{1} + T_1\| \, \|\mathbb{1} + T_2\|$, and we are done. □

It turns out however that, unlike the classical Lovász function, our definition (5) is not multiplicative. In fact, it fails even for tensoring certain channels with a trivial channel. (A channel is called *trivial* if it maps all states ρ on A to a constant state σ_0 on B, thus having associated operator subspace $\mathcal{L}(A)$, which corresponds to the complete non-commutative graph.)

We shall show that one may even take the identity channel $\mathrm{id} : \mathcal{L}(\mathbb{C}^d) \to \mathcal{L}(\mathbb{C}^d)$, which has subspace $S = \mathbb{C}\mathbb{1}_d$. We claim that

$$\sup_n \vartheta\big(\mathbb{1}_d \otimes \mathcal{L}(\mathbb{C}^n)\big) = d^2. \tag{6}$$

Proof First we show that the value d^2 can be attained with $n = d$. Namely, for any orthogonal operator basis of unitaries, including the identity, $\mathbb{1} = U_0, U_1, \ldots U_{d^2-1}$, let

$$T = \sum_{\alpha=1}^{d^2-1} U_\alpha \otimes \overline{U}_\alpha = d^2 \Phi_d - \mathbb{1} \otimes \mathbb{1},$$

so that $\mathbb{1} \otimes \mathbb{1} + T$ is, up to a normalisation factor of d^2, the maximally entangled state Φ_d. Since the latter is positive semidefinite, and

$$T \in \big(\mathbb{1}_d \otimes \mathcal{L}(\mathbb{C}^d)\big)^\perp = \mathbb{1}_d^\perp \otimes \mathcal{L}(\mathbb{C}^d),$$

because all U_α are traceless, we obtain indeed $\vartheta\big(\mathbb{1}_d \otimes \mathcal{L}(\mathbb{C}^d)\big) \geq d^2$.

Second, it remains to show that for all n, $\vartheta\big(\mathbb{1}_d \otimes \mathcal{L}(\mathbb{C}^n)\big) \leq d^2$. For this consider any $T \in \mathbb{1}_d^\perp \otimes \mathcal{L}(\mathbb{C}^n)$ such that $\mathbb{1}_d \otimes \mathbb{1}_n + T \geq 0$. On the one hand, clearly $\mathrm{Tr}_d(\mathbb{1}_d \otimes \mathbb{1}_n + T) = d\mathbb{1}_n$ – which has norm d –, on the other hand, it is well-known that the partial trace over a d-dimensional system can change the operator norm (in fact, any p-norm) by at most a factor of d [14]. Thus, $\|\mathbb{1}_d \otimes \mathbb{1}_n + T\| \leq d^2$. □

This motivates the following better definition, a kind of norm completion of ϑ:

Definition 5 *Observing that* $\big(S \otimes \mathcal{L}(\mathbb{C}^n)\big)^\perp = S^\perp \otimes \mathcal{L}(\mathbb{C}^n)$, *let the* quantum Lovász function *be*

$$\begin{aligned} \widetilde{\vartheta}(S) &:= \sup_n \vartheta\big(S \otimes \mathcal{L}(\mathbb{C}^n)\big) \\ &= \sup_n \max\big\{\|\mathbb{1} + T\| : T \in S^\perp \otimes \mathcal{L}(\mathbb{C}^n),\ \mathbb{1} + T \geq 0\big\}, \end{aligned} \tag{7}$$

where the supremum is over all integers n, and the maximum in the second line is again over Hermitian operators T.

Note that by our above result on the ideal channel, also $\widetilde{\vartheta}(S) \leq |A|^2$. And for classical graphs G, since ϑ is multiplicative and $\mathcal{L}(\mathbb{C}^n)$ is the operator space version of the complete graph, $\widetilde{\vartheta}(S) = \vartheta(S) = \vartheta(G)$.

Remark From the point of view of operator spaces it might appear rather natural and pleasing that we have to consider the norm completion $\widetilde{\vartheta}$, so to speak, of ϑ, by taking the supremum over tensor products with arbitrary full matrix spaces.

There seems to be an analogy to the construction of the completely bounded norm from the "naive" norm of operator maps [29]. Much like completely bounded norms [33], also our definition via completion will turn out to be given by a semidefinite programme (see the next section).

From the definition it is clear that in general $\widetilde{\vartheta}$ inherits the supermultiplicativity from ϑ:

Lemma 6 *$\widetilde{\vartheta}$ is supermultipicative, i.e. for non-commutative graphs $S_1 < \mathcal{L}(A_1)$ and $S_2 < \mathcal{L}(A_2)$,*

$$\widetilde{\vartheta}(S_1 \otimes S_2) \geq \widetilde{\vartheta}(S_1)\widetilde{\vartheta}(S_2).$$

□

More importantly, however, it is related to the entanglement-assisted independence number:

Lemma 7 *For any non-commutative graph S, $\widetilde{\alpha}(S) \leq \widetilde{\vartheta}(S)$. Furthermore, $\widetilde{\vartheta}$ is monotonic with respect to subgraphs, i.e. when $S \subset S'$ for two non-commutative graphs, then $\widetilde{\vartheta}(S) \geq \widetilde{\vartheta}(S')$.*

Proof The monotonicity is inherited from ϑ. For the relation to $\widetilde{\alpha}$, the argument is an extension of the one we made for the unassisted case and $\vartheta(S)$. Namely, recall that we may pad the channel by a sufficiently large dummy register that goes into a trivial channel, and find a state $\rho \in \mathcal{S}(A \otimes R)$ and unitaries U_m on $A \otimes R$ such that for $1 \leq m \neq m' \leq N$,

$$U_m \rho U_{m'}^\dagger \in S^\perp \otimes \mathcal{R}.$$

Evidently this is unchanged under rescaling ρ, so we replace it by a multiple X with largest eigenvalue 1: $X = |\varphi\rangle\langle\varphi| + X'$, where $X' \perp |\varphi\rangle\langle\varphi|$ is a rest which satisfies $\|X'\| \leq 1$. Now we consider the candidate

$$T = \sum_{m \neq m'} U_m X U_{m'}^\dagger \otimes |m\rangle\langle m'| \in S^\perp \otimes \mathcal{L}(R \otimes \mathbb{C}^n).$$

This is an eligible operator in eq. (7) because

$$\begin{aligned}
\mathbb{1} + T &= \mathbb{1} + \sum_{m \neq m'} U_m X U_{m'}^\dagger \otimes |m\rangle\langle m'| \\
&\geq \sum_{mm'} U_m X U_{m'}^\dagger \otimes |m\rangle\langle m'| \\
&= \left(\sum_m U_m \sqrt{X} \otimes |m\rangle \right) \left(\sum_{m'} \sqrt{X} U_{m'}^\dagger \otimes \langle m'| \right) = MM^\dagger \geq 0,
\end{aligned}$$

where in the second line we have used $\mathbb{1} \geq \sum_m U_m X U_m^\dagger \otimes |m\rangle\langle m|$.

Finally, to bound the norm, define the unit vector $|\phi\rangle = \frac{1}{\sqrt{N}} \sum_m U_m |\varphi\rangle \otimes |m\rangle$. Then observe

$$\|\mathbb{1} + T\| \geq \|MM^\dagger\| \geq \langle\phi|MM^\dagger|\phi\rangle = N,$$

which completes the proof. □

V. SEMIDEFINITE FORMULATION AND OTHER PROPERTIES

We shall now simplify the expression for $\widetilde{\vartheta}$, putting an a priori limit on the dimension n of the extension system. Namely, for fixed n, we have

$$\|\mathbb{1} \otimes \mathbb{1} + T\| = \max_{|\phi\rangle} \langle\phi|(\mathbb{1} \otimes \mathbb{1} + T)|\phi\rangle,$$

for $T \in S^\perp \otimes \mathcal{L}(\mathbb{C}^n)$, $\mathbb{1} \otimes \mathbb{1} + T \geq 0$, where the maximum is over unit vectors in $A \otimes \mathbb{C}^n$. Now we can use a trick analogous to Lovász' [26, Theorem 4]: with the maximally entangled vector $|\Phi\rangle = \sum_{i=1}^{|A|} |i\rangle^A |i\rangle^{A'}$ there exists an operator $M : A' \to \mathbb{C}^n$ such that $|\phi\rangle = (\mathbb{1} \otimes M)|\Phi\rangle$. Thanks to $\mathrm{Tr}_A\, \Phi = \mathbb{1}_{A'}$, the normalisation of $|\phi\rangle$ translates into $\rho = M^\dagger M$ being a state (i.e. of trace 1) on A'. Thus,

$$\langle\phi|(\mathbb{1} \otimes \mathbb{1} + T)|\phi\rangle = \langle\Phi|\big(\mathbb{1} \otimes \rho + (\mathbb{1} \otimes M^\dagger)T(\mathbb{1} \otimes M)\big)|\Phi\rangle,$$

and the crucial observation is that $T' = (\mathbb{1} \otimes M^\dagger)T(\mathbb{1} \otimes M) \in S^\perp \otimes \mathcal{L}(A')$. As a consequence, we have proved

Theorem 8 *For any non-commutative graph $S < \mathcal{L}(A)$,*

$$\begin{aligned} \widetilde{\vartheta}(S) = \max\ & \langle\Phi|(\mathbb{1} \otimes \rho + T')|\Phi\rangle \\ \text{s.t. }\ & T' \in S^\perp \otimes \mathcal{L}(A'), \quad \mathrm{Tr}\,\rho = 1, \\ & \mathbb{1} \otimes \rho + T' \geq 0, \quad \rho \geq 0, \end{aligned} \tag{8}$$

which is a semidefinite characterisation of $\widetilde{\vartheta}$. □

This has two important consequences: first, we have now an optimisation with a *bounded* dimension of the extension (namely $|A|$) and furthermore it is semidefinite [32], so it is computable efficiently. Second, and much deeper, we have a dual semidefinite programme for the same value that is a minimisation problem and allows us to put upper bounds on $\widetilde{\vartheta}(S)$.

Theorem 9 *The dual of the semidefinite programme (8) gives*

$$\begin{aligned} \widetilde{\vartheta}(S) = \min\ & \|\mathrm{Tr}_A\, Y\| \\ \text{s.t. }\ & Y \in S \otimes \mathcal{L}(A'), \quad Y \geq \Phi, \end{aligned} \tag{9}$$

where A' is isomorphic to A.

Before we prove this, we record an immediate corollary:

Corollary 10 *$\widetilde{\vartheta}$ is multiplicative: for non-commutative graphs $S_1 < \mathcal{L}(A_1)$ and $S_2 < \mathcal{L}(A_2)$,*

$$\widetilde{\vartheta}(S_1 \otimes S_2) = \widetilde{\vartheta}(S_1)\widetilde{\vartheta}(S_2). \tag{10}$$

Indeed, we know already that it is supermultiplicative, so we only have to show that it is also submultiplicative. I.e., for two subspaces $S_i < \mathcal{L}(A_i)$, $\widetilde{\vartheta}(S_1 \otimes S_2) \leq \widetilde{\vartheta}(S_1)\widetilde{\vartheta}(S_2)$. But that we can read off from the dual: if Y_1 is dual feasible for S_1 and Y_2 for S_2, then clearly $Y_1 \otimes Y_2$ is dual feasible for $S_1 \otimes S_2$. At the same time, $\|\mathrm{Tr}_{A_1A_2}\, Y_1 \otimes Y_2\| = \|(\mathrm{Tr}_{A_1}\, Y_1) \otimes (\mathrm{Tr}_{A_2}\, Y_2)\| = \|(\mathrm{Tr}_{A_1}\, Y_1)\|\, \|(\mathrm{Tr}_{A_2}\, Y_2)\|$, and we are done. □

Proof (of Theorem 9) The primal is a semidefinite programme of the general form

$$\max\ \operatorname{Tr} CX \ \text{ s.t. } \ \ell(X) = \underline{b},\ X \geq 0,$$

with a linear vector-valued function $\ell : \mathcal{L}(H)_{\text{sa}} \to \mathbb{R}^n$. The dual of such a form is given by

$$\min\ \underline{b}^\top \cdot \underline{y} \ \text{ s.t. } \ \ell'(\underline{y}) \geq C,$$

where $\ell' : \mathbb{R}^n \to \mathcal{L}(H)_{\text{sa}}$ is the adjoint linear map to ℓ [32].

In the present case, let $d = |A|$; the matrix X will be

$$X = \begin{bmatrix} X_{11} & * \\ * & X_{22} \end{bmatrix} = \begin{bmatrix} \rho & * \\ * & \mathbb{1} \otimes \rho + T' \end{bmatrix},$$

and the linear constraint has to ensure this form of the matrix, the trace normalisation of X_{11} and the fact that T' is orthogonal to $S \otimes \mathcal{L}(A')$. The objective function is given by $C = \begin{bmatrix} 0 & 0 \\ 0 & \Phi \end{bmatrix}$.

Thus, fixing an operator basis $\{F_\alpha\}_\alpha$ of S, and a basis $\{G_\beta\}_\beta$ of $\mathcal{L}(A')$, the components of ℓ are

$$\ell_0(X) = \operatorname{Tr} X_{11} = \operatorname{Tr} X \begin{bmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{bmatrix} =: \operatorname{Tr} X L_0,$$

$$\ell_{\alpha\beta}(X) = \operatorname{Tr}(F_\alpha \otimes G_\beta)(X_{22} - \mathbb{1} \otimes X_{11}) = \operatorname{Tr} X \begin{bmatrix} (-\operatorname{Tr} F_\alpha) G_\beta & 0 \\ 0 & F_\alpha \otimes G_\beta \end{bmatrix} =: \operatorname{Tr} X L_{\alpha\beta},$$

while $b_0 = 1$ and all other $b_{\alpha\beta} = 0$. With these notations, the adjoint map ℓ' can be constructed as

$$\ell'(\underline{y}) = y_0 L_0 + \sum_{\alpha\beta} y_{\alpha\beta} L_{\alpha\beta}.$$

Using that the second term in ℓ' is a generic element of $S \otimes \mathcal{L}(A')$, we can simplify our expressions, and find that the objective function is y_0, and that

$$\ell'(\underline{y}) = \begin{bmatrix} Y_{11} & 0 \\ 0 & Y_{22} \end{bmatrix}, \ \text{ where}$$
$$Y_{22} \in S \otimes \mathcal{L}(A'),$$
$$Y_{11} = y_0 \mathbb{1} - \operatorname{Tr}_A Y_{22}.$$

I.e., the constraints are $Y_{22} \geq \Phi$ and $y_0 \mathbb{1} \geq \operatorname{Tr}_A Y_{22}$, proving the form of the dual, since the optimal y_0 is the norm (maximum eigenvalue) of $\operatorname{Tr}_A Y_{22}$.

To finish, we only need to verify feasibility of both primal and dual; for the primal this is shown by $T' = 0$, for the dual by $Y = d\mathbb{1} \otimes \mathbb{1}$. Thus, the conditions of strong duality are fulfilled, both primal and dual optimal values are finite and equal. $\square$

Using this dual, we can now show that $\widetilde{\vartheta}$ is monotonic under pre- and post-processings of the channel defining S, and more generally under enlarging the graph and going to induced subgraphs.

Corollary 11 *For non-commutative graphs $S < \widehat{S}$, $\widetilde{\vartheta}(S) \geq \widetilde{\vartheta}(\widehat{S})$.*

For a non-commutative graph S, let $U : A_0 \hookrightarrow A$ be an isometry and consider the induced subgraph $S' = U^\dagger S U < \mathcal{L}(A_0)$. Then, $\widetilde{\vartheta}(S') \leq \widetilde{\vartheta}(S)$.

As a consequence, let $S = \mathcal{N}^*(\mathcal{L}(C))$ with a channel $\mathcal{N} : \mathcal{L}(A) \to \mathcal{L}(B)$. Then, $\widetilde{\vartheta}$ is non-increasing when going to non-commutative graphs obtained by either pre- or post-processing $\mathcal{N}$.

Proof For a larger graph $\widehat{S} > S$, we know already $\widetilde{\vartheta}(\widehat{S}) \leq \widetilde{\vartheta}(S)$; and post-processing gives exactly rise to a larger graph $\widehat{S} > S$.

Induced subgraphs are more interesting: Let $Y \in S \otimes \mathcal{L}(A')$ be an optimal solution of the dual semidefinite programme for $\widetilde{\vartheta}(S)$, according to Theorem 9, i.e. $Y \geq \Phi_{AA'}$ and $\| \operatorname{Tr}_A Y \| = \widetilde{\vartheta}(S)$. But then $Y' = (U^\dagger \otimes \mathbb{1})Y(U \otimes \mathbb{1}) \in S' \otimes \mathcal{L}(A')$, $Y' \geq \Phi_{A_0 A_0'}$ and $\| \operatorname{Tr}_{A_0} Y' \| \leq \| \operatorname{Tr}_A Y \|$.

Finally, any graph $S' < \mathcal{L}(A')$ originating from a pre-processing of $\mathcal{N}$ is obtained from an isometry $U : A' \hookrightarrow A \otimes D$, via $S' = U^\dagger (S \otimes \mathcal{L}(D))U$. But S and $S \otimes \mathcal{L}(D)$ have the same $\widetilde{\vartheta}$, by definition, and since S' is an induced subgraph of the latter, we are done. □

We end this section by remarking that the dual in Theorem 9 simplifies considerably in the case of classical channels, i.e. $S = \operatorname{span}\{|x\rangle\langle x'| : x = x' \text{ or } x \sim x'\} < \mathcal{L}(\mathbb{C}X)$. Note that $|\Phi\rangle \in A \otimes A'$ is invariant under unitaries of the form $U \otimes \overline{U}$, and that S is stabilised by diagonal unitaries $Z = \sum_x e^{i\varphi_x} |x\rangle\langle x|$. Hence, with every dual feasible Y, we get an equally good dual feasible solution $(Z \otimes \overline{Z})Y(Z \otimes \overline{Z})^\dagger$, so by the triangle inequality, we can find a dual optimal solution among the operators invariant under conjugation with $Z \otimes \overline{Z}$, i.e. $Y = \sum_{xx'} Y_{xx'} |xx\rangle\langle x'x'|$. The constraints are $Y \geq \Phi = \sum_{xx'} |xx\rangle\langle x'x'|$ and $Y_{xx'} = 0$ if $x \not\sim x'$, while the objective function is the norm of the partial trace $\operatorname{Tr}_A Y = \sum_x Y_{xx} |x\rangle\langle x|$. Thus, we arrive at

Corollary 12 *For a classical graph G, Lovász' ϑ is given by the semidefinite programme*

$$\vartheta(G) = \min \left\{ \max_{x \in X} Y_{xx} : Y \in S,\ Y \geq J \right\},$$

where S is the non-commutative graph associated to G, meaning $Y_{xx'} = 0$ whenever $x \not\sim x'$, and J is the all-1 matrix. □

VI. APPLICATIONS AND DISCUSSION

There are a few immediate consequences, the most obviously important being obtained by putting together Lemma 7 and Corollary 10:

Corollary 13 *For any non-commutative graph $S < \mathcal{L}(A)$, $C_{0E}(S) \leq \log \widetilde{\vartheta}(S)$.* □

Then, for a classical channel with confusability graph G, we observed earlier that $\widetilde{\vartheta}(S) = \vartheta(G)$. Hence, $\widetilde{\alpha}(G) \leq \vartheta(G)$ and so:

Corollary 14 *For any graph G, $C_{0E}(G) \leq \log \vartheta(G)$.* □

This answers an open question from [12], which is nontrivial because there it is shown that $\widetilde{\alpha}(G)$ may be strictly larger than $\alpha(G)$.

E.g., we can now compute the entanglement-assisted zero-error capacity of the "Bell-Kochen-Specker" channels discussed in [12]. These are all disjoint unions of n copies of K_d, with some extra edges between the complete components, such that G is exactly the orthogonality graph of a set of nd vectors in $\mathbb{C}^d$. If the set of vectors gives rise to a Kochen-Specker proof of non-contextuality, this means $\alpha(G) \leq n-1$. On the other hand, in [12] it is shown that $\widetilde{\alpha}(G) \geq n$, using a rank-$d$ maximally entangled state $\frac{1}{d}\Phi_d$. Here, we can now see $n \leq \widetilde{\alpha}(G) \leq \vartheta(G) \leq n$, as shown by the dual feasible solution $Y = n \bigoplus \Phi_d$, which has $\| \operatorname{Tr}_A Y \| = n$. Thus, $\widetilde{\alpha}(G) = \vartheta(G) = n$ and we also learn that $C_{0E}(G) = \log n$. (One could, however, see this also directly by noting that G contains a disjoint union of n complete graphs as a subgraph.)

While we do not have a separating upper bound for the unassisted capacity $C_0(G)$ of these graphs, of course even as a bound on the independence number, our Corollary 13 is an improvement over Lovász [26], since we find that $\vartheta(G)$ is even larger or equal than $\widetilde{\alpha}$. In this sense, the increase of independence number from α to $\widetilde{\alpha}$ due to entanglement-assistance somehow "explains" the fact that Lovász' ϑ is not always a tight bound [20] – and in fact, it is quite possible that $C_{0E}(G)$ can be strictly larger than $C_0(G)$.

There are, furthermore, other quantum channels for which $\widetilde{\vartheta}(S) = \vartheta(S)$. For instance, perhaps the simplest one is $S = \Delta^\perp$, where $S^\perp = \mathbb{C}\Delta$ (with a traceless Hermitian operator Δ) is one-dimensional. In that case one has evidently $\widetilde{\alpha}_U(S) = \widetilde{\alpha}(S) = \widehat{\alpha}(S) = 2$ (see Proposition 2). In fact, the gap between $\widehat{\alpha}(S)$ and $\widetilde{\vartheta}(S)$ can be made arbitrarily large, since the latter can be up to d as shown by the example of

$$\Delta = \begin{bmatrix} d-1 & & & \\ & -1 & & \\ & & \ddots & \\ & & & -1 \end{bmatrix},$$

and in fact similar examples show that every real value between 2 and d is realised as some $\widetilde{\vartheta}(\Delta^\perp)$. We do not know a better upper bound on $C_{0E}(S)$ for this channel other than $\log \widetilde{\vartheta}(S)$.

Non-commutative graphs for which we can determine C_{0E} include all $S < \mathcal{L}(\mathbb{C}^2)$:

- If $S = \mathbb{C}\mathbb{1}$, then the channel is perfect, and by superdense coding we can achieve $\widetilde{\alpha}(S) = 4 = \widetilde{\vartheta}(S)$, hence $C_{0E}(S) = 2$.
- The other extreme is $S = \mathcal{L}(\mathbb{C}^2)$, then $\widetilde{\vartheta}(S) = 1$ and hence $C_{0E} = 0$.
- In the intermediate case, $2 \leq \dim S \leq 3$, and we claim $C_{0E}(S) = 1$. Indeed, the capacity is largest for the smallest subspace, hence we consider only $\dim S = 2$. The subspace is spanned by $\mathbb{1}$ and another operator, which we may take to be diagonal and traceless, thus w.l.o.g. Z. This is the subspace corresponding to the noiseless classical (i.e. Z-dephasing) channel $\mathcal{N}(\rho) = \sum_{b=0,1} |b\rangle\langle b|\rho|b\rangle\langle b|$, which clearly has entanglement-assisted capacity 1, even in the Shannon setting [6], which can be achieved error-free and without entanglement since $\alpha(S) = 2$, $C_0(S) = 1$. For $\dim S = 3$, we still have $\widetilde{\alpha}(S) = \widetilde{\alpha}_U(S) = 2$, by Proposition 2.

Yet another one can be found in [15, Thm. 3, eq. (8)], where a channel is constructed with non-commutative graph $S = \mathbb{1}_2 \otimes \mathbb{1}_d + \mathbb{1}^\perp \otimes \mathcal{L}(\mathbb{C}^d)$, so that $S^\perp = \mathbb{1}_2 \otimes \mathbb{1}_d^\perp$. It was shown that $\widetilde{\alpha}(S) \geq d^2$, and indeed, because S contains $\mathcal{L}(\mathbb{C}^2) \otimes \mathbb{1}_d$, $\widetilde{\vartheta}(S) \leq \widetilde{\vartheta}(\mathcal{L}(\mathbb{C}^2) \otimes \mathbb{1}_d) = d^2$, hence $C_{0E}(S) = 2\log d$.

Perhaps the most interesting open question regarding the entanglement-assisted zero-error capacity is whether $C_{0E}(S) = \log \widetilde{\vartheta}(S)$. Note that this would imply that C_{0E} is multiplicative (whereas C_0 is not [1]); one might recall that entanglement-assistance has made also the theory of communication via quantum channels more elegant [6], and likewise so-called *XOR games* [8], for which a semidefinite characterisation lead to multiplicativity of the optimal winning probability. A most challenging test case is presented by the above non-commutative graphs $S = \Delta^\perp$, for which we do not even know $\widetilde{\alpha}(S \otimes S)$ at the time of writing, nor in fact $\widehat{\alpha}(S \otimes S)$. Does it perhaps hold that $\widehat{C}_{0E}(S) \leq \log\left(1 + \dim S^\perp\right)$ in general? – which by the above examples would imply a separation between $\log \widetilde{\vartheta}(S)$ and $C_{0E}(S)$.

Another question pertains to a possible generalisation of a property of Lovász' $\vartheta(G)$: Is it true that $\widetilde{\vartheta}(S_1 \cap S_2) \leq \widetilde{\vartheta}(S_1)\widetilde{\vartheta}(S_2)$? Note that it holds for classical graphs – because the intersection is an induced subgraph of the strong product along the diagonal –, and that it would be an extension of the multiplicativity statement.

Third, it is a bit unsatisfactory that we have three entanglement-assisted independence numbers. Are they really different? Is it perhaps true that at least they lead to the same asymptotic capacities? What is in general the relation between $\widehat{\alpha}(S)$ and $\widetilde{\vartheta}(S)$?

Finally, looking back at our path, it may seem odd and in fact a bit arbitrary that we arrived at a Lovász type bound on the entanglement-assisted independence number. Do there exist similar bounds for the unassisted zero-error capacity and the zero-error quantum capacity that are strictly better than $\widetilde{\vartheta}(S)$?

VII. NON-COMMUTATIVE GRAPH THEORY?

In this last section, no longer concerned with zero-error communication but driven by the idea of developing a proper theory of non-commutative graphs, we will finally give the proper definition of graphs, of subgraphs, induced substructures, etc. For this purpose, we have to come back to the characterisation of S in terms of the adjoint $\widehat{\mathcal{N}}^*$ of the complementary channel (Lemma 1). Such maps, by being completely positive and unital, obey the Kadison-Schwarz (operator) inequality

$$\widehat{\mathcal{N}}^*(X)^\dagger \widehat{\mathcal{N}}^*(X) \leq \widehat{\mathcal{N}}^*(X^\dagger X),$$

for all $X \in \mathcal{L}(C)$. The set of operators which satisfy this with equality is, by Choi's theorem [7, 29], the so-called *multiplicative domain*

$$\mathcal{M} := \big\{ X \in \mathcal{L}(C) \text{ s.t. } \forall Y\ \widehat{\mathcal{N}}^*(X)\widehat{\mathcal{N}}^*(Y) = \widehat{\mathcal{N}}^*(XY) \big\}, \tag{11}$$

which is in fact a $*$-subalgebra (containing $\mathbb{1}$) of $\mathcal{L}(C)$, and restricted to it, $\widehat{\mathcal{N}}^* : \mathcal{M} \to S_0 := \widehat{\mathcal{N}}^*(\mathcal{M})$ is a $*$-algebra homomorphism. The image S_0 is clearly a subspace of S, a $*$-algebra itself, and by eq. (11) it satisfies

$$S_0 S = S S_0 = S,$$

i.e., S is a (left and right) S_0-module, all presented explicitly as operator subspaces of $\mathcal{L}(A)$. In fact, it is even a so-called *Hilbert-S_0-module* [25]; all we need is to choose an S_0-valued inner product $\langle \cdot, \cdot \rangle : S \times S \to S_0$, which we shall however always assume to be defined on $\mathcal{L}(A)$. The inner product should be linear in the first, and conjugate linear in the second element, $\langle X, Y \rangle^* = \langle Y, X \rangle$, it should respect the module structure (from the right) as $\langle X, Ya \rangle = \langle X, Y \rangle a$ for $X, Y \in S$ and $a \in S_0$ (which is equivalent to $\langle Xa, Y \rangle = a^\dagger \langle X, Y \rangle$), and $\langle X, X \rangle \geq 0$ with equality iff $X = 0$. This defines a very strong notion of orthogonality in $\mathcal{L}(A)$.

In the first part of the paper, we effectively treated every non-commutative graph as if it had trivial $S_0 = \mathbb{C}\mathbb{1}$. In this case, there is a whole family of inner products $\langle X, Y \rangle = (\mathrm{Tr}\, X^\dagger R Y S)\mathbb{1}$ for some positive definite $0 < R, S \in \mathcal{L}(A)_{\text{sa}}$, but there are many more. Because of its importance for the independent set question discussed above, and its relation to matrix multiplication, we assign special status to the Hilbert-Schmidt inner product (i.e. $R = S = \mathbb{1}$), which shall be the default when no inner product is specified.

To obtain some more structure, note that since, $S_0 < S$ it is reasonable to demand that $\langle \mathbb{1}, Y \rangle = Y$ and $\langle X, \mathbb{1} \rangle = X^\dagger$ for $X, Y \in S_0$ (which is equivalent to asking $\langle X, Y \rangle = X^\dagger Y$). Motivated by this, and using also the left module structure, we could ask for the even stronger property $\langle X, aY \rangle = \langle a^\dagger X, Y \rangle$ for $a \in S_0$ (together with $\langle \mathbb{1}, \mathbb{1} \rangle = \mathbb{1}$).

In general, the structure theorem for finite dimensional $*$-algebras implies

$$S_0 = \bigoplus_{j=1}^{r} \mathcal{L}(A_j) \otimes \mathbb{1}_{Z_j}, \quad \text{with} \quad A = \bigoplus_{j=1}^{r} A_j \otimes Z_j,$$

while $\langle \cdot, \cdot \rangle$ gives rise to a conditional expectation $E(X) = \langle \mathbb{1}, X\rangle$ (satisfying $E(XA) = E(X)A$ for $X \in S$ and $A \in S_0$ and $E(A) = A$, by the above additional assumption). The general form of the conditional expectation is

$$E(X) = \bigoplus_{j=1}^{r} \left[\operatorname{Tr}_{Z_j} (P_j \otimes \sqrt{\zeta_j}) X (P_j \otimes \sqrt{\zeta_j}) \right] \otimes \mathbb{1}_{Z_j},$$

with the projectors $P_j = \mathbb{1}_{A_j}$ onto A_j and $Q_j = \mathbb{1}_{Z_j}$ onto Z_j, and states $\zeta_j \in \mathcal{S}(Z_j)$. For the conditional expectation to be *faithful*, i.e. $X \geq 0$ and $E(X) = 0$ implying $X = 0$, it is necessary and sufficient that all the ζ_j are faithful.

Now, from the left and right module structure,

$$\begin{aligned} S = \bigoplus_{j,k=1}^{r} (P_j \otimes Q_j) S (P_k \otimes Q_k), \ \ &\text{and for each } j,\ k, \\ (P_j \otimes Q_j) S (P_k \otimes Q_k) &= (\mathcal{L}(A_j) \otimes Q_j) S (\mathcal{L}(A_k) \otimes Q_k) \\ &= \mathcal{L}(A_k \to A_j) \otimes S_{jk}, \end{aligned} \tag{12}$$

where $S_{jk} < \mathcal{L}(Z_k \to Z_j)$ such that $S_{kj} = S_{jk}^{\dagger}$ and $Q_j \in S_{jj}$. From this we see that each non-commutative graph gives rise to an underlying classical graph "skeleton"

$$G(S_0 < S) := \big(V = [r], E = jk : S_{jk} \neq 0\big).$$

A general inner product is not uniquely defined by its conditional expectation, but each conditional expectation E gives rise to the following canonical inner product

$$\langle X, Y \rangle_E = E(X^{\dagger} Y) = \bigoplus_{j=1}^{r} \left[\operatorname{Tr}_{Z_j} (P_j \otimes \sqrt{\zeta_j}) X^{\dagger} Y (P_j \otimes \sqrt{\zeta_j}) \right] \otimes \mathbb{1}_{Z_j}. \tag{13}$$

As before for the Hilbert-Schmidt inner product, the tracial states $\zeta_j = \frac{1}{|Z_j|} \mathbb{1}_{Z_j}$ are distinguished because of the symmetry of the resulting inner product

$$E(UXU^{\dagger}) = UE(X)U^{\dagger}, \ \text{ for unitaries } U \text{ s.t. } US_0U^{\dagger} = S_0,$$

which characterises them uniquely. (And hence its relation to the usual matrix product.) This choice is understood as the default if we only specify S_0 but not an inner product.

We did not need all this additional structure before, but it motivates our eventual definition:

Definition 15 *A* non-commutative graph *is a pair $S_0 < S$ of operator subspaces of some $\mathcal{L}(A)$, with a complex Hilbert space A, equipped with an inner product $\langle X, Y \rangle$ that makes S a Hilbert left and right S_0-module.*

That is, S_0 is a $*$-subalgebra of $\mathcal{L}(A)$ containing $\mathbb{1}$ and contained in S, $S = S^{\dagger}$ and S is a left and right S_0-module with respect to matrix multiplication, i.e. $S_0 S = S S_0 = S$. The inner product satisfies $\langle X, Y \rangle^* = \langle Y, X \rangle$ for all $X, Y \in \mathcal{L}(A)$, $\langle \mathbb{1}, \mathbb{1} \rangle = \mathbb{1}$, $\langle X, Ya \rangle = \langle X, Y \rangle a$ for $X, Y \in S$ and $a \in S_0$ (which *is equivalent to $\langle Xa, Y\rangle = a^\dagger\langle X, Y\rangle)$, $\langle X, aY\rangle = \langle a^\dagger X, Y\rangle$, for $a \in S_0$, and $\langle X, X\rangle \geq 0$, with equality iff $X = 0$.*

In fact, with the conditional expectation $E(X) = \langle \mathbb{1}, X\rangle$ from $\mathcal{L}(A)$ to S_0, we shall only look at inner products of the form $\langle X, Y\rangle = E(X^\dagger Y)$.

To emphasise the dependence on S_0 and E, we shall call S a (non-commutative) S_0-graph *(if we don't specify the inner product), or more precisely an E-*graph.

We do not have a sufficient overview over the literature to claim that this concept is entirely new and unexplored. The term – apart from a single occurrence in the context of non-commutative geometry [18] – appears not to have been used before. And while there is some literature regarding finitely generated Hilbert-modules over finite-dimensional $*$-algebras, to the best of our knowledge no-one seems to ever have made the connection to graph theory.

Remark Abstractly, there seems no reason to insist on $\mathcal{N}$ being trace preserving, which would correspond to S not necessarily containing the identity matrix $\mathbb{1}$. In the first part of the paper we could indeed have relaxed the definition of non-commutative graph to be an operator space $S = S^\dagger < \mathcal{L}(A)$ containing some positive definite element $D > 0$.

However, the above concepts do not go well with this generalisations, as for non-unital $\widehat{N}^*$ we do not have unital $*$-subalgebra structure of S_0, nor is it characterised by Choi's theorem. Thus we stick with our original definition for now, leaving an exploration of alternative definitions for later.

Clearly, the same operator subspace S can originate from different channels, which might however have different S_0. All pairs $S_0 < S$ according to the above definition occur, however. The $*$-subalgebra S_0 serves as a kind of "diagonal" in the operator space S, in fact, whereas S generalises the edges of a graph, S_0 is representative of the vertices (their number being remembered in the dimension $|A|$ of the underlying Hilbert space). It is perhaps helpful to remember, for the sake of intuition, to recall one of the original motivations to consider Hilbert modules [21] as an abstract version of vector bundles over manifolds, represented as the module of vector fields over the algebra of continuous functions, where the inner product originates from a Riemannian structure of the vector bundle; this intuition has been immensely fruitful in the creation of non-commutative geometry and its applications [9].

The basic example of course is once more the classical graph: we saw before that starting from a noisy channel, one can arrive at the confusability graph in its non-commutative guise

$$S = \mathrm{span}\{|x\rangle\langle x'| : x = x' \text{ or } x \sim x'\} < \mathcal{L}(\mathbb{C}X),$$

but that made no distinction between vertices ($x = x'$) and proper edges. Looking at the quantum version of the channel as discussed in section II, one can see that S_0 will contain all $|x\rangle\langle x|$. By appropriately modifying the channel N, for instance by considering $N' = \frac{1}{2}N \oplus \frac{1}{2}\mathrm{id}_X$ (with the same input alphabet X and the larger output alphabet $X \,\dot\cup\, Y$), one can indeed enforce $S_0 = \mathrm{span}\{|x\rangle\langle x| : x \in X\}$, with the canonical conditional expectation $\mathrm{diag}(X) = \sum_x |x\rangle\langle x|X|x\rangle\langle x|$. Now, it is clear that one can recover the graph G from $S_0 < S$ up to isomorphism. Thus, the classical graphs are precisely the diag-graphs, the graph structure recovered precisely as the skeleton $G(S_0 < S)$ of the algebraic data.

Furthermore, the module S over diag is generated by a single element (using left and right multiplication), for instance by the Laplacian of the graph. This property is shared by all non-commutative graphs where in eq. (12), S_{jk} is at most one-dimensional for all j, k.

A class of examples that are already more "quantum" are graphs $S < \mathcal{L}(A)$, with $S_0 = \mathbb{C}\mathbb{1}$ and a conditional expectation of the form $E_\rho(X) = \mathrm{Tr}(\rho X)\mathbb{1}$ for a state $\rho \in \mathcal{S}(A)$. Such a non-commutative graph we call *ρ-graph*, and all we require for it is $\mathbb{1} \in S = S^\dagger$.

Going back once more to the motivation of our concepts from channels, one may recall that each classical channel $N : X \to Y$ also gives rise to a *bipartite* graph with vertex set $X \stackrel{\cup}{\cdot} Y$, where $x \in X$ and $y \in Y$ are connected by an edge iff $N(y|x) > 0$. This bipartite graph captures much more about the channel than the confusability graph, and indeed Shannon's zero-error feedback result [31] and Cubitt *et al.*'s regarding assistance by non-signalling resources [12] can be formulated in terms of this bipartite graph. As its quantum version we propose to consider, for a quantum channel $\mathcal{N} : \mathcal{L}(A) \to \mathcal{L}(B)$ with $\mathcal{N}(\rho) = \sum_j E_j \rho E_j^\dagger$, the operator subspace

$$Z := \operatorname{span}\{E_j\} < \mathcal{L}(A \to B). \tag{14}$$

This space was crucial in the proof of Proposition 2, and it is evident that the non-commutative graph of the channel is obtained as $S = Z^\dagger Z < \mathcal{L}(A)$. Furthermore, one can confirm that indeed Z is still a right S_0-module, and that the above conditional expectation E makes it indeed a Hilbert module, via the same rule $\langle X, Y\rangle = E(X^\dagger Y)$ for $X, Y \in Z$.

Proof This is essentially only an extension of Choi's reasoning [7]. We use the Stinespring representation of the channel, with isometry $V : A \hookrightarrow B \otimes C$, such that $\widehat{\mathcal{N}}^*(m) = V^\dagger(\mathbb{1}_B \otimes m_C)V$.

Then, a generic element of Z can be written $X = (\mathbb{1}_B \otimes \langle\xi|_C)V = \sum_j \xi_j E_j$ for an appropriate vector $|\xi\rangle \in C$. A generic element of S_0 instead is $a = \widehat{\mathcal{N}}^*(m)$, for an element $m \in \mathcal{M} < \mathcal{L}(C)$ of the multiplicative domain. We wish to show that $Xa \in Z$, and indeed we will find that

$$Xa = (\mathbb{1}_B \otimes \langle\xi'|_C)V, \ \text{ with } |\xi'\rangle = m^\dagger|\xi\rangle.$$

First, noting $Xa = (\mathbb{1}_B \otimes \langle\xi|_C)VV^\dagger(\mathbb{1}_B \otimes m_C)V$, Choi's theorem tells us

$$\begin{aligned} a^\dagger X^\dagger X a &= V^\dagger(\mathbb{1}_B \otimes m_C^\dagger)VV^\dagger(\mathbb{1}_B \otimes |\xi\rangle\langle\xi|_C)VV^\dagger(\mathbb{1}_B \otimes m_C)V \\ &= V^\dagger(\mathbb{1}_B \otimes m^\dagger|\xi\rangle\langle\xi|_C m)V = V^\dagger(\mathbb{1}_B \otimes |\xi'\rangle\langle\xi'|_C)V, \end{aligned}$$

hence $Xa = (U_B^{X,a} \otimes \langle\xi'|_C)V$ for some unitary $U^{X,a} \in \mathcal{U}(B)$. But for another $Y \in Z$, $b \in S_0$, once more by Choi's theorem,

$$\begin{aligned} b^\dagger Y^\dagger X a &= V^\dagger(\mathbb{1}_B \otimes n_C^\dagger)VV^\dagger(\mathbb{1}_B \otimes |\upsilon\rangle\langle\xi|_C)VV^\dagger(\mathbb{1}_B \otimes m_C)V \\ &= V^\dagger(\mathbb{1}_B \otimes n^\dagger|\upsilon\rangle\langle\xi|_C m)V, \end{aligned}$$

showing $U^{X,a} = U^{Y,b}$ for all X, Y and a, b, which concludes the proof. □

This motivates the following definition, for which each cptp map yields an example:

Definition 16 *A* non-commutative (directed) bipartite graph *with "vertex spaces" A and B is a subspace $Z < \mathcal{L}(A \to B)$ together with a unital $*$-subalgebra $S_0 < S = Z^\dagger Z < \mathcal{L}(A)$, and a conditional expectation $E : \mathcal{L}(A) \to S_0$, such that Z is a right S_0-module, and indeed a Hilbert module for the inner product $\langle X, Y\rangle = E(X^\dagger Y)$.*

Again, all non-commutative bipartite graphs originate from some cptp channel.

We call an E-graph $S < \mathcal{L}(A)$ and an E'-graph $S' < \mathcal{L}(A')$ *isomorphic*, if there exists a unitary isomorphism U between A and A' such that

$$USU^\dagger = S', \quad US_0U^\dagger = S_0', \quad \text{and} \quad UE(X)U^\dagger = E'(UXU^\dagger).$$

This implies a definition of automorphism, too, and we denote the automorphism group of S as $\operatorname{Aut}(S) < \mathcal{U}(A)$. We say that the automorphism group *acts (vertex) transitively* if the only operators in the commutant of S_0 that also commute with the automorphism group, are $\mathbb{C}\mathbb{1}$.

Now we can start defining the usual graph notions: we call a *complete graph* a pair $S_0 < S = \mathcal{L}(A)$ together with any faithful conditional expectation $E : \mathcal{L}(A) \to S_0$. To be precise, it is the complete E-graph.

The *complement* of an E-graph S is defined to be the subspace

$$S_0 < S^c := S_0 + S^{(\perp_E)} = S_0 + \{X \in \mathcal{L}(A) : \forall Y \in S \ \langle X, Y\rangle_E = 0\},$$

which by virtue of the Hilbert-module property is again an E-graph. The definition is made in such a way that $(S^c)^c = S$ and $S \cap S^c = S_0$; in particular the complete E-graph is the complement of S_0 (which we call the *empty* E-graph). Note that the notion of complement depends on the conditional expectation E and its image S_0.

Also *graph products* are defined easily: and E-graph $S < \mathcal{L}(A)$ (with subalgebra S_0) and an E'-graph $S' < \mathcal{L}(A')$ (with subalgebra S_0') give rise to the (strong) product, which is the $E \otimes E'$-graph $S \otimes S' < \mathcal{L}(A \otimes A')$, with subalgebra $S_0 \otimes S_0'$. Thus we also have the powers $S^{\otimes n}$, which are $E^{\otimes n}$-graphs.

The *disjoint union* of an E-graph $S < \mathcal{L}(A)$ and an E'-graph $S' < \mathcal{L}(A')$ is the direct sum $S \oplus S' < \mathcal{L}(A \oplus A')$ (with subalgebra $S_0 \oplus S_0'$). Denoting the projections onto A and A' in $A \oplus A'$ by P and $P' = \mathbb{1} - P$, this is an $E \oplus E'$-graph, where $(E \oplus E')(X) := E(PXP) \oplus E'(P'XP')$. If the graphs originate from channels, their direct sum originates from the direct sum channel. Note that the corresponding orthogonal sectors in the direct sum are always perfectly distinguishable; one can make them indistinguishable by adding the full operator sets $\mathcal{L}(A \to A')$ and $\mathcal{L}(A' \to A)$ to the direct sum, "filling up the off-diagonal blocks":

$$S \boxplus S' := S \oplus S' + \mathcal{L}(A \to A') + \mathcal{L}(A' \to A) < \mathcal{L}(A \oplus A'),$$

which we call the *complete union* of the graphs (because it corresponds to placing a complete bipartite graph between the vertex spaces A and A').

Clearly, products, disjoint and complete unions are associative, and both unions are distributive with respect to the graph product.

Proposition 17 *Both ϑ and $\widetilde{\vartheta}$ are additive under disjoint unions:*

$$\vartheta(S \oplus S') = \vartheta(S) + \vartheta(S'), \quad \widetilde{\vartheta}(S \oplus S') = \widetilde{\vartheta}(S) + \widetilde{\vartheta}(S').$$

Furthermore, α is additive, and $\widetilde{\alpha}$ and $\widehat{\alpha}$ are superadditive under disjoint unions:

$$\alpha(S \oplus S') = \alpha(S) + \alpha(S'), \quad \widetilde{\alpha}(S \oplus S') \geq \widetilde{\alpha}(S) + \widetilde{\alpha}(S'), \quad \widehat{\alpha}(S \oplus S') \geq \widehat{\alpha}(S) + \widehat{\alpha}(S').$$

Finally, all $f \in \{\alpha, \widetilde{\alpha}, \widehat{\alpha}, \vartheta, \widetilde{\vartheta}\}$ satisfy the following identity:

$$f(S \boxplus S') = \max\{f(S), f(S')\}.$$

Proof We only need to show the first claim for ϑ. By eq. (5),

$$\vartheta(S \oplus S') = \max\left\{\left\| \begin{bmatrix} \mathbb{1} + T & M \\ M^\dagger & \mathbb{1} + T' \end{bmatrix} \right\| : T \in S^\perp,\ T' \in S'^\perp \right\},$$

where the maximum is restricted to positive semidefinite block matrices. It is an easy observation that for all positive semidefinite L_1, L_2,

$$\max_M \left\{ \left\| \begin{bmatrix} L & M \\ M^\dagger & L' \end{bmatrix} \right\| : \begin{bmatrix} L & M \\ M^\dagger & L' \end{bmatrix} \geq 0 \right\} = \|L\| + \|L'\|,$$

from which the assertion follows.

The independence numbers are clearly superadditive, so it is left to show that $\alpha(S \oplus S') \leq \alpha(S) + \alpha(S')$. For this, let $\{|\phi_m\rangle : m = 1, \ldots, N\}$ be an independent set of $S \oplus S'$, so that for all $m \neq m'$,

$$(S \oplus S')^\perp \ni |\phi_m\rangle\langle\phi_{m'}| = (P \oplus P')|\phi_m\rangle\langle\phi_{m'}|(P \oplus P'),$$

which is equivalent to

$$P|\phi_m\rangle\langle\phi_{m'}|P \in S^\perp \quad \text{and} \quad P'|\phi_m\rangle\langle\phi_{m'}|P' \in S'^\perp.$$

Thus, up to normalisation, the set $\mathcal{A} = \{m : P|\phi_m\rangle \neq 0\}$ gives rise to an independent set in S, and likewise $\mathcal{B} = \{m : P'|\phi_m\rangle \neq 0\}$ for S'. Because each m is in at least one of $\mathcal{A}$ or $\mathcal{B}$, the claim follows.

Finally, the $\boxplus$-$\max$-identities follow almost immediately from $(S \boxplus S')^\perp = S^\perp \oplus S'^\perp$. □

Another easy notion is the *distance-$\leq t$-graph* of an E-graph $S < \mathcal{L}(A)$: this is the subspace $S^t = S \cdot S \cdots S$ (the t-fold product), which is indeed an E-graph. By convention, here $S^0 := S_0$.

It may happen that the same S is an E-graph (with subalgebra S_0) and an F-graph (with subalgebra $S_1 > S_0$), such that E factors through F, i.e. there is a conditional expectation $G : S_1 \to S_0$ such that $E = G \circ F$. We call then the F-graph $S_1 < S$ a *refinement* of the E-graph $S_0 < S$. (The idea being that with F and S_1, the graph has more vertices.) Conversely, by concatenating the conditional expectation E with another one $E' : S_0 \to S_1 < S_0$, we can obtain *coarse grainings* of any E-graph as $E' \circ E$-graphs.

The notions of subgraph and induced subgraph are more subtle, because we have to take care of the conditional expectation. The simplest is when S' is a *proper subgraph* of an E-graph S, which means that $S_0 < S' < S$ and that S' is a sub-Hilbert-S_0-module of S with the same inner product: $S_0 S' = S' S_0 = S'$. We call proper subgraphs also *E-subgraphs*. Less strict, we call an E'-graph S' with subalgebra S_0' a (generally: improper) *subgraph* of the E-graph $S < \mathcal{L}(A)$ if $S' < S$ and $S_0' < S_0$, and $E'|_{S'} = E|_{S'}$

Induced subgraphs of an E-graph $S < \mathcal{L}(A)$ (with algebra S_0) are defined with respect to a subspace $A' < A$ with projector P: the E'-graph $S' := PSP < \mathcal{L}(A')$ (with algebra S_0') is called *proper induced subgraph* if $PS_0P = S_0'$ and the restriction to A' commutes with the conditional expectations: $E'(PXP) = PE(X)P$ for all $X \in \mathcal{L}(A)$. Again, there is a less strict notion of *induced subgraph*, which only demands $PS_0P < S_0'$ and that $S_0' < S'$ is a refinement of $PS_0P < S'$.

To illustrate these notions, we note that the Stinespring dilation theorem implies that every E-graph $S_0 < S < \mathcal{L}(A)$ is a proper induced subgraph of the strong product between a complete F-graph $\mathcal{L}(C)$ an empty ρ-graph $\mathbb{C}\mathbb{1} < \mathcal{L}(B)$. We can also re-interpret the independence numbers of a non-commutative graph S as the largest dimensions of (improper) induced subgraphs: an induced empty ρ-graph $\mathbb{C}\mathbb{1} < \mathcal{L}(A')$ for $\alpha_q(S)$ [and such A' we should hence call a *quantum independent set*], and an induced diag-graph $\mathrm{span}\{|m\rangle\langle m| : m = 1, \ldots, |A'|\} < \mathcal{L}(A')$ for $\alpha(S)$ [and such A' we should call an *independent set*]. Cliques are defined analogously.

Going back to eq. (12), recall that a non-commutative graph S has the form

$$\bigoplus_{j=1}^{r} \mathcal{L}(A_j) \otimes \mathbb{1}_{Z_j} = S_0 < S = \bigoplus_{j,k=1}^{r} \mathcal{L}(A_k \to A_j) \otimes S_{jk}.$$

From this we can construct the graph $\widetilde{S}_0$,

$$\bigoplus_{j=1}^{r} \mathbb{C}\mathbb{1}_{Z_j} =: \widetilde{S}_0 < \widetilde{S} := \bigoplus_{j,k=1}^{r} S_{jk},$$

which is an induced subgraph of S over a *commutative* diagonal $\widetilde{S}_0$. In addition, S itself is an induced subgraph of $\widetilde{S} \otimes \mathcal{L}(R)$ for large enough $|R|$. One can think of S as obtained from $\widetilde{S}$ by "blowing up the vertices": each vertex becomes a complete graph $K_{|A_j|}$, and each edge a complete bipartite graph $K_{|A_k|,|A_j|}$. Because of these relations, S and $\widetilde{S}$ share the values of α, $\widetilde{\alpha}$. $\widehat{\alpha}$ and $\widetilde{\vartheta}$ (though not of ϑ).

As yet, we do not have many illuminating examples of non-commutative graphs, nor can we offer applications to classical graph theory. Instead, we close with highlighting several questions motivated by the above definitions.

- Algorithmic consequences: Non-commutative graph isomorphism is at least as hard as classical graph isomorphism, but are they of the same order? Similarly, graph non-isomorphism has efficient interactive proofs, does this extend to non-commutative graphs? Finally, induced substructures such as independent sets are NP-complete for classical graphs, and QMA-complete for non-commutative graphs (again for independent sets) – but is it still QMA-complete for quantum independent sets? Or for entanglement-assisted independent sets? An interesting question in particular is, whether one can put a priori bounds on the dimension of the entangled state referred to in the definitions for $\widetilde{\alpha}$ and $\widehat{\alpha}$.

- For classical graphs on n vertices, the largest known ratio between independence number and Lovász function occurs for random graphs and is $\Omega(\sqrt{n}/\log n)$, which is conjectured to be maximal. What is the largest value of $\widetilde{\vartheta}(S)/\widetilde{\alpha}(S)$ for non-commutative graphs? (Our example $S = \Delta^{\perp}$ in section VI shows a lower bound of $|A|/2$.)

- Random graphs are a powerful tool in combinatorics; what would be the natural non-commutative random graphs? The simplest one can think of is to fix the dimension D of a subspace $S = S^{\dagger} < \mathcal{L}(\mathbb{C}^n)$ containing $\mathbb{1}$, and to choose it uniformly at random according to the Haar-induced measure on the Grassmannian (very much like what is done in [11]). What are the expected values of clique and independence numbers, and of our $\widetilde{\vartheta}$ as functions of n and D?

- The bipartite graphs $Z < \mathcal{L}(A \to B)$ play a central role in the zero-error capacity of classical channels assisted by feedback or non-signalling correlation, as we have mentioned. Does this extend to quantum channels in the appropriate sense? For this, one first has to confirm that the classical noiseless feedback-assisted zero-error capacity, $C_{0F}(\mathcal{N})$, can be expressed in terms of Z alone. This is indeed possible, even when the feedback is allowed to be an arbitrary quantum message after each channel use. We are currently exploring fractional packing/covering numbers for non-commutative bipartite graphs, with the motivation of extending Shannon's zero-error capacity theory to quantum channels with feedback.

- There are many other graph notions we didn't generalise yet: Perhaps the most interesting ones are chromatic number and perfectness of a graph. Is there a Laplacian operator with distinguished properties in each non-commutative graph? Finally, is there a good notion of edge contraction which would lead to a theory of graph minors?

Remark A final comment on the definition $\widetilde{\vartheta}(S)$: There, it would seem more natural to consider the subspace orthogonal with respect to the Hilbert-module inner product $\langle \cdot, \cdot \rangle_E$ (in particular excluding the entire diagonal S_0 from $S^{(\perp_E)}$). This highlights the dependence of the notion of orthogonality on the inner product chosen. In our definition of the independence numbers – and then again when we defined ϑ, $\widetilde{\vartheta}$ – we relied on the underlying Hilbert space structure, which led us to consider the Hilbert-Schmidt inner product on $\mathcal{L}(A)$, and more generally the conditional expectations with tracial ζ_j. This seems to suggest that there are privileged conditional expectations to define the Hilbert module. We leave an investigation of this issue to future explorations of non-commutative graphs.

Acknowledgments

It is a pleasure to thank many people for discussions and feedback on the present work, including Toby Cubitt, Debbie Leung, Will Matthews, Ashley Montanaro, Tomasz Paterek, Marcin Pawłowski, and Aram Harrow.

While completing this paper, we learned of a direct proof by Salman Beigi [4] that the entanglement-assisted independence number of a classical channel (and hence a classical graph) is bounded by Lovász' ϑ (Corollary 14). We are grateful to him for sharing his manuscript with us prior to publication.

RD is partly supported by QCIS, University of Technology, Sydney, and the NSF of China (Grant Nos. 60736011 and 60702080). SS is supported by a Newton International Fellowship. AW is supported by the European Commission, the U.K. EPSRC, the Royal Society and a Philip Leverhulme Prize. The Centre for Quantum Technologies is funded by the Singapore Ministry of Education and the National Research Foundation as part of the Research Centres of Excellence programme.

---

[1] N. Alon, "The Shannon capacity of a union", Combinatorica **18**(3):301-310 (1998).

[2] N. Alon, E. Lubetzky, "The Shannon Capacity of a Graph and the Independence Number of its Powers", IEEE Trans. Inf. Theory **52**(5):2172-2176 (2006).

[3] J. Barrett, N. Linden, S. Massar, S. Pironio, S. Popescu, D. Roberts, "Nonlocal correlations as an information-theoretic resource", Phys. Rev. A **71**:022101 (2005).

[4] S. Beigi, "Entanglement-assisted zero-error capacity is upper bounded by the Lovász theta function", `arXiv[quant-ph]:1002.2488` (2010).

[5] S. Beigi, P. W. Shor, "On the Complexity of Computing Zero-Error and Holevo Capacity of Quantum Channels", `arXiv[quant-ph]:0709.2090` (2007).

[6] C. H. Bennett, P. W. Shor, J. A. Smolin, A. V. Thapliyal, "Entanglement-assisted classical capacity of noisy quantum channels", Phys. Rev. Lett. **83**(15):3081-3084 (1999); "Entanglement-assisted capacity of a quantum channel and the reverse Shannon theorem", IEEE Trans. Inf. Theory **46**(10):2637-2655 (2002).

[7] M.-D. Choi, "A Schwarz inequality for positive linear maps on C*-algebras", Illinois J. Math. **18**:565-574 (1974).

[8] R. Cleve, W, Slofstra, F. Unger, S. Upadhyay, "Strong Parallel Repetition Theorem for Quantum XOR Proof Systems", `arXiv:quant-ph/0608146` (2006).

[9] A. Connes, *Noncommutative Geometry*, Academic Press, 1994.

[10] I. Csiszár, J. Körner, *Information Theory: Coding Theorems for Discrete Memoryless Systems*, Academic Press, New York, 1982.

[11] T. S. Cubitt, J. Chen, A. W. Harrow, "Superactivation of the Asymptotic Zero-Error Classical Capacity of a Quantum Channel", `arXiv[quant-ph]:0906.2547` (2009).

[12] T. S. Cubitt, D. W. Leung, W. Matthews, A. Winter, "Improving zero-error classical communication with entanglement", arXiv[quant-ph]:0911.5300 (2009).

[13] T. S. Cubitt, G. Smith, "Super-Duper-Activation of Quantum Zero-Error Capacities", `arXiv [quant-ph]:0912.2737` (2009).

[14] W. van Dam, P. Hayden, "Rényi-entropic bounds on quantum communication", `arXiv:quant-ph/0204093` (2002).

[15] R. Duan, "Super-Activation of Zero-Error Capacity of Noisy Quantum Channels", `arXiv[quant-ph]:0906.2527` (2009).
[16] R. Duan, Y. Shi, "Entanglement between Two Uses of a Noisy Multipartite Quantum Channel Enables Perfect Transmission of Classical Information", Phys. Rev. Lett. **101**:020501 (2008).
[17] E. G. Effros, Z.-J. Ruan, *Operator Spaces*, Oxford University Press, Oxford, New York, 2000.
[18] T. Filk, "Connes Distance Function for Commutative and Noncommutative Graphs", Int. J. Theor. Phys. **39**(2):223-230 (2000).
[19] F. Guo, Y. Watanabe, "On graphs in which the Shannon capacity is unachievable by finite product", IEEE Trans. Inf. Theory **36**(3):622-623 (1990).
[20] W. Haemers, "On Some Problems of Lovbz Concerning the Shannon Capacity of a Graph", IEEE Trans. Inf. Theory **25**(2):231-232 (1979); "An upper bound for the Shannon capacity of a graph", Coll. Math. Soc. J. Bolyai **25**:267-272 (1978).
[21] I. Kaplansky, "Modules over operator algebras", Amer. J. Math. **75**:839-853 (1953).
[22] E. Knill, R. Laflamme, "Theory of quantum error-correcting codes", Phys. Rev. A **55**(2):900-911 (1997).
[23] D. Knuth, "The Sandwich Theorem", Electr. J. Comb. **1**(1):A1 (1994).
[24] J. Körner, A. Orlitsky, "Zero-Error Information Theory", IEEE Trans. Inf. Theory **44**(6):2207-2229 (1998).
[25] E. C. Lance, *Hilbert C^*-Modules: A toolkit for operator algebraists*, LMS Lecture Notes Series 210, Cambridge University Press, Cambridge, 1995.
[26] L. Lovász, "On the Shannon Capacity of a Graph", IEEE Trans. Inf. Theory **25**(1):1-7 (1979).
[27] R. A. C. Medeiros, R. Alleaume, G. Cohen, F. M. de Assis, "Zero-error capacity of quantum channels and noiseless subsystems", VI Int. Telecommunications Symposium (ITS), 3-6 Sept 2006, Fortaleza CE, Brazil (2006); R. A. C. Medeiros, R. Alleaume, G. Cohen, F. M. de Assis, "Quantum states characterization for the zero-error capacity", `arXiv:quant-ph/0611042` (2006).
[28] S. Pironio, M. Navascués, A. Acín, "Convergent relaxations of polynomial optimization problems with non-commuting variables", `arXiv[math.OC]:0903.4368` (2009); M. Navascués, S. Pironio, A. Acín, "A convergent hierarchy of semidefinite programs characterizing the set of quantum correlations", New J. Phys. **10**:073013 (2008).
[29] V. Paulsen, *Completely Bounded Maps and Operator Algebras*, Cambridge Studies in Advanced Mathematics **78**, Cambridge University Press, 2003.
[30] C. E. Shannon, "A mathematical theory of communication", Bell Syst. Tech. J. **27**:379-423 & 623-656 (1948).
[31] C. E. Shannon, "The zero-error capacity of a noisy channel", IRE Trans. Inform. Theory, **IT-2**(3):8-19 (1956).
[32] L. Vandenberghe, S. Boyd, "Semidefinite Programming", SIAM Review **38**(1):49-95 (1996).
[33] J. Watrous, "Semidefinite programs for completely bounded norms", `arXiv[quant-ph]:0901.4709` (2009).