Superradiance and Phase Multistability in Circuit Quantum Electrodynamics

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We show that it is possible to observe superradiance and phase multistability in superconducting circuit quantum electrodynamics (QED) using state-of-the-art experiments. It is demonstrated that superradiant microwave pulses can be observed for small numbers of qubits in the presence of energy relaxation and non-uniform qubit-field coupling strengths. This paves the way for circuit QED implementations of superradiant state readout and decoherence free subspace state encoding in subradiant states. The system considered here also exhibits phase multistability at large field driving amplitudes, which have possible applications in collective qubit measurement and quantum feedback.

Superradiance is the collective spontaneous emission from an ensemble of N highly correlated two level systems (qubits) coupled to one or more electrmagnetic field modes [1]. The collective emission intensity, which scales with $\propto N^2$, is of fundamental interest in quantum optics and has been the subject of many theoretical and experimental works [2–4]. With greater control over a range of quantum systems, it is now possible to observe superradiance in quantum dots [5], Bose Einstein condensates [6] and nitrogen vacancy centers in diamond [7]. Furthermore superradiance and the related phenomena of subradiance have quantum information based applications such as decoherence free subspace state encoding in subradiant states [8] and superradiant state readout [9].

There are two major obstacles to observing superradiance from an ensemble of Nqubits: short qubit coherence times and weak and inhomogeneous coupling to the field. In practice, large ensembles are required to overcome these obstacles as the peak intensity of the emitted radiation scales quadratically with the ensemble size. Superradiant phenomena are yet to be detected in small ensembles due to weak non-uniform qubit-field coupling rates and the high degree of control required to isolate the system from environmental decoherence.

In this Letter we show that it is possible to observe small ensemble superradiance in a superconducting circuit quantum electrodynamics (QED) system. Circuit-QED is an ideal platform to observe small ensemble superradiance due to small qubit dephasing, large qubit-field coupling rate compared to the rate of qubit energy relaxation, and uniform qubit-field coupling rates [10]. Another phenomenon that can be observed using the same system, with the addition of coherent driving, is phase multistability. Phase multistability describes the phenomena where the qubit-resonator system in the steady state is well approximated by a set of coupled, driven and damped harmonic oscillators [11].

Here, we study several transmon qubits coupled to the field of a transmission line resonator (TLR) (Fig. 1). In the frame rotating at angular frequency ω the dynamics of the system are described by the master equation,

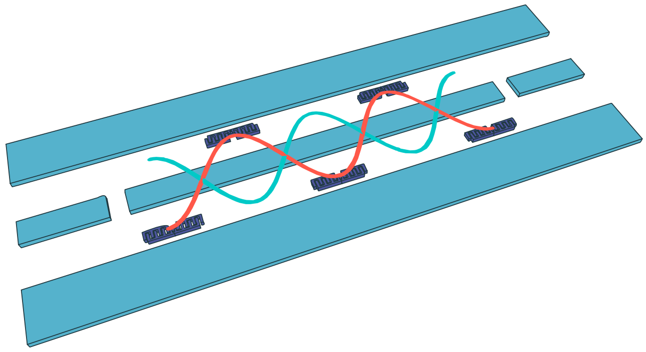

FIG. 1: (Color online) Five transmon qubits (dark blue) coupled to the quantized field of a TLR (sinusoids).

$$\dot{\rho} \;=\; -i[H,\rho] + \frac{\kappa}{2}\mathcal{D}[a]\rho + \sum_{j=1}^{N}\frac{\gamma_j^s}{2}\mathcal{D}[\sigma_j^-]\rho + \frac{\gamma_j^p}{2}\mathcal{D}[\sigma_j^z]\rho \quad (1)$$

where γ_j^s and γ_j^p are for the jth qubit the energy relaxation and dephasing rate respectively, κ is the resonator decay rate, $\mathcal{D}[A]\rho = 2A\rho A^\dagger - A^\dagger A\rho - \rho A^\dagger A$ and $\hbar = 1$. The qubit-resonator system evolves in the interaction picture under the Tavis-Cummings Hamiltonian [12],

$$H = \Delta_r a^\dagger a + \sum_{j=1}^{N}\left(\frac{\Delta_{q,j}}{2}\sigma_j^z + g_{N,j}(\sigma_j^- a^\dagger + a\sigma_j^+)\right), \quad (2)$$

where, the jth qubit couples to the field at the rate $g_{N,j}$, $\Delta_{q,j} = \omega_{q,j} - \omega$ is the detuning from the jth qubit transition frequency $\omega_{q,j}$ and $\Delta_r = \omega_r - \omega$ is the detuning from the resonator frequency ω_r.

Superradiance: In order to obtain an analytic solution for the intensity of emitted photons from the TLR, it is necessary to make several assumptions. First, we assume that each qubit couples identically to the field mode ($g_{N,j} \approx \bar{g}_N$) and has the same transition frequency ($\omega_{q,j} = \omega_q$, $\Delta_q = \omega_q - \omega$). This can be achieved by optimizing qubit position and tuning the transition frequency via independent flux lines to each qubit [10, 13]. Secondly, we assume that the field in the resonator decays at a faster rate than the qubits, $\kappa \gg \bar{g}_N \gg \gamma_j^s, \gamma_j^p$. This bad-cavity limit ensures the superradiant pulse will escape the resonator and not be re-absorbed by the qubits. Lastly, as the qubit-field coupling rate, $\bar{g}_N$, is typically two orders of magnitude larger than the energy relaxation rate [10], it is assumed $\gamma_j^s \approx 0$. In addition, the dephasing rate for transmon qubits are negligible $\gamma_j^p \approx 0$. The validity of these assumptions will be tested against the numerical solution of (1) in the latter part of this Letter.

As the resonator field decays on a time scale much faster than that of the qubits we can adiabatically eliminate the field. The master equation for the reduced density operator describing the qubits only is given by,

$$\dot{\rho}_q = -i[\frac{\Delta_q}{2}J_z - \Delta_r \frac{\bar{g}_N^2}{|\Gamma|^2}J_+J_-, \rho_q] + \frac{\kappa}{2}\frac{\bar{g}_N^2}{|\Gamma|^2}\mathcal{D}[J_-]\rho_q, \quad (3)$$

where, $\Gamma = \kappa/2 + i\Delta_r$, and it is assumed that the resonator is initially empty. The collective operators are defined as, $J_i = \sum_{j=1}^N \sigma_j^i$, where $i = \{+, -, z\}$, and σ_j^i denote the individual qubit Pauli matrices. Equation (3) corresponds to the superradiance master equation (SRME) that was derived for atomic ensembles, after a suitable parameter substitution [2, 7, 14].

We now seek an expression for the intensity of photons escaping the TLR, $I_N(t)$. Expanding (3) in the Dicke basis [1], the probability that the system is in one of the Dicke states $|l, m\rangle$ is, $P(l, m, t) = \langle l, m|\rho_q(t)|l, m\rangle$, where, $\mathbf{J}^2|l, m\rangle = l(l + 1)|l, m\rangle$, $J_z|l, m\rangle = 2m|l, m\rangle$ and $N/2 \geq l \geq |m| \geq 0$. Using the SRME (3) the population rates follow [15],

$$\begin{aligned}\dot{P}(l, m, t) &= \frac{\kappa\bar{g}_N^2}{|\Gamma|^2}[(l - m)(l + m + 1)P(l, m + 1, t) \\ &- (l + m)(l - m + 1)P(l, m, t)]. \quad (4)\end{aligned}$$

Here, we consider the initial condition that all qubits are excited, i.e., $P(l, m, 0) = P(N/2, N/2, 0)$, although this approach can easily be extended to take into account other initial conditions [16]. As l is conserved by the SRME we introduce the variable, $n = l - m = 0, 1, ..., N$, which corresponds to the number of photons emitted from the resonator when the system is in the initial state $m = N/2$. Equation (4) can now be rewritten,

$$\begin{aligned}\dot{P}(n, \tau) &= (N - n + 1)nP(n - 1, \tau) \\ &- (N - n)(n + 1)P(n, \tau). \quad (5)\end{aligned}$$

where we have rescaled the time $\tau = \gamma t$ and $\gamma = \kappa\bar{g}_N^2/|\Gamma|^2$. To proceed, we Laplace transform (5), subject to the full excitation initial condition, $P(n, 0) = \delta_{n,0}$,

$$\begin{aligned}sQ(n, s) - \delta_{n,0} &= (N - n + 1)nQ(n - 1, s) \\ &- (N - n)(n + 1)Q(n, s), \quad (6)\end{aligned}$$

where, $Q(n, s)$ is the Laplace transform of $P(n, \tau)$. From (6) we find $Q(0, s) = 1/(s + N)$. Continuing recursively we find,

$$Q(n > 0, s) = \frac{1}{s + N}\prod_{i=1}^{n}\frac{(N - i + 1)i}{s + (N - i)(i + 1)}. \quad (7)$$

Upon inverting the transform (7) we obtain the populations of the Dicke states, $P(n, \tau)$ [15]. The intensity of photons emitted from the TLR can be found from the Dicke state populations using, $I_N(\tau) = \frac{\partial}{\partial\tau}\sum_n nP(n, \tau)$. The system is superradiant if the maximum intensity, I_N^{max}, is greater than N, the initial intensity of N independent qubits [7].

We now consider the experimentally relevant systems of TLRs with $N = 3, 4$ or 5 qubits. To find the probability that the system is in one of the Dicke states, $|N/2, N/2 - n\rangle$, we invert the transform (7) to obtain $P(n, t)$ and hence the intensities,

$$\begin{aligned}I_3(\tau) &= 3e^{-3\tau}(12\tau - 7) + 24e^{-4\tau}. \quad (8a)\\ I_4(\tau) &= (72\tau + 96)e^{-6\tau} + 4e^{-4\tau}(36\tau - 23), \quad (8b)\\ I_5(\tau) &= \frac{5}{3}[162e^{-9\tau} + 16e^{-8\tau}(24\tau - 1) \\ &+ e^{-5\tau}(240\tau - 143)]. \quad (8c)\end{aligned}$$

As $I_3^{max} \approx 3.2 > 3$, $I_4^{max} \approx 4.9 > 4$, and, $I_5^{max} \approx 6.9 > 5$, these systems exhibit superradiance. In Fig. 2 the intensity from the numerical solution of the original master equation (1) is compared to equations (8a-c) for experimentally feasible parameters. The approximate solutions (8a-c) closely resemble the results from the original master equation despite the assumptions, $\kappa \gg \bar{g}_N \gg \gamma_j^s, \gamma_j^p$,

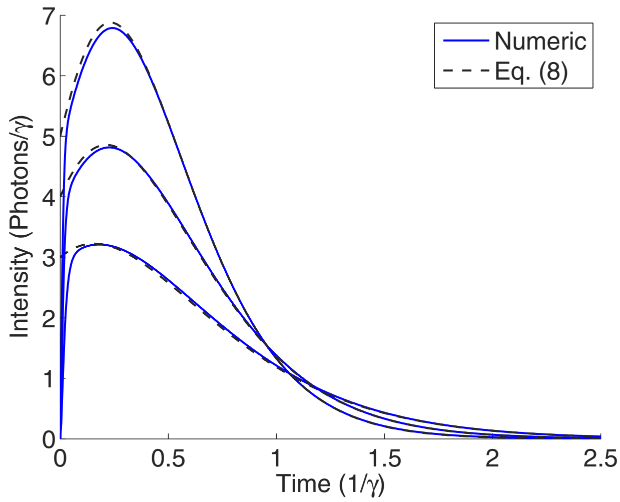

FIG. 2: Intensity of radiation from the resonator for $N = 3, 4, 5$ (in order of peak height lowest to highest). Parameters used are $g_{3,j}/2\pi = (83.7, 85.7, 85.1)$ MHz [10], $g_{4,j}/2\pi = (69.4, 69.1, 68.6, 69.7)$ MHz, $g_{5,j}/2\pi = (59.0, 59.4, 59.9, 60.9, 60.7)$ MHz, and $(\kappa, \gamma_j^s, \gamma_j^p, \Delta_q, \Delta_r)/2\pi = (2000, 0.19, 0, 0, 0)$ MHz.

$g_{N,j} \approx \bar{g}_N$ and $\gamma_j^s \approx 0$. The two solutions converge for larger κ, identical qubit-field coupling rates, and $\gamma_j^s \approx 0$. For each system the superradiant peak is large enough to be easily resolved using existing detection schemes [17].

The deviations from the numerical solution are a result of the violation of the assumptions on κ, g and γ_j^s. Initially there are no photons escaping the TLR ($I_N(0) = 0$) as the field is in vacuum. As the qubits transfer energy to the TLR, the finite decay rate κ traps the photons in the resonator for a short period leading to the delay of the superradiant pulse. Non-uniform coupling rates, $\bar{g}_N$, and energy relaxation, γ_j^s, lead to a slower decay rate due to coupling to different co-operation numbers, $l < N/2$ [14]. Furthermore, energy relaxation reduces the number of photons emitted in the resonator mode.

The above results demonstrate that it is possible to observe small sample superradiance in a TLR in the presence of energy relaxation and non-uniform coupling strengths. Superradiance can also be observed for $N > 5$ qubits, provided $\kappa \gg \bar{g}_N \gg \gamma_j^s$ and $g_{N,j} \approx \bar{g}_N$. For a TLR with resonance frequency ω_r, the length determines the number of qubits as each is placed at field antinodes. As the coupling strength $\bar{g}_N$ is inversely related to the mode volume [18], the coupling is reduced for large numbers of qubits. As $\bar{g}_N \to \gamma_j^s$, losses due to energy relaxation and coupling to different co-operation numbers will dominate the dynamics and superradiant effects will become difficult to observe. Therefore the upper bound on the number of qubits is governed by the ratio $\bar{g}_N/\gamma_j^s$. Nevertheless, the proposal herein allows the experimental study of small sample superradiance and may have applications in quantum information, including decoherence free subspace state encoding using subradiant states [8] and superradiant state readout [9].

Phase Multistability: Returning to (1) in the strong coupling regime ($\bar{g}_N > \kappa, \gamma_j^s, \gamma_j^p$) we consider the effect of strong driving of the TLR, $H \to H_{\mathcal{E}} = H + i\mathcal{E}(a^\dagger - a)$ where, $\mathcal{E}$ is the drive amplitude and it is assumed $\Delta_q = \Delta_r = 0$. In this regime, the phenomenon of phase multistability can be observed; where in the steady state, the combined qubit-TLR system is well approximated by several coupled, driven and damped harmonic oscillators [11]. In the single qubit case ($g \equiv g_{1,1}, \gamma^s \equiv \gamma_1^s$) it is well known that for $2\mathcal{E} > g > \kappa > \gamma^s$ the system exhibits phase bistability [19–21]. Phase bistability, observed in cavity QED only recently [20], has been the basis for several proposals including nanophotonic switching [20], quantum feedback and qubit measurement [22]. However, phase multistability for $N > 1$ is yet to be observed.

Phase bistability can be described in the $\gamma^s \to 0$ limit as follows: at large intra-cavity photon numbers, n, the difference in energy between successive Jaynes-Cummings (JC) manifolds $|n, \pm\rangle \leftrightarrow |n+1, \pm\rangle$ is $E_{n+1}^\pm - E_n^\pm \approx \omega_r \pm g/2\sqrt{n}$, where $|n, \pm\rangle = 1/\sqrt{2}(|n+1, g\rangle \pm |n, e\rangle)$. Strong driving of the TLR at ω_r will populate large n JC eigenstates and cause transitions along two separate pathways, $|n, \pm\rangle \leftrightarrow |n+1, \pm\rangle$. The driving is detuned from the transition frequency of each path by the manifold dependent detuning, $\pm g/(2\sqrt{n})$. Furthermore, the steady state density matrix of the system can be approximated by a mixture of two uncoupled, damped harmonic oscillators, which are driven off-resonantly [19, 21]. As $t \to \infty$ the coherent state amplitude of each oscillator $|\alpha_{ss}^\pm\rangle$ is $\alpha_{ss}^\pm = f(2\mathcal{E}f \pm ig)/\kappa$, where $f = \sqrt{1 - (g/2\mathcal{E})^2}$. The two coherent states $|\alpha_{ss}^\pm\rangle$ can be detected by reconstructing the Q-function after homodyne measurement of the field.

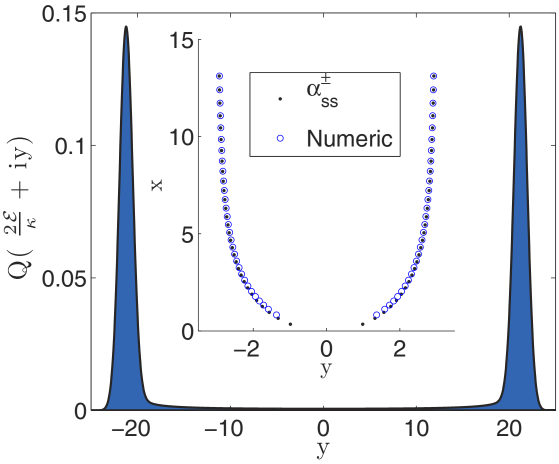

FIG. 3: The Q-function in Eq. (9) at $x = 2\mathcal{E}/\kappa$ with parameters $(g, \kappa, \gamma^s)/2\pi = (85, 4, 0.19)$ MHz. *Inset:* The location of the two peaks in the steady state Q-function for a range of driving strengths $\mathcal{E}/2\pi = 45 - 200$ MHz, increasing from bottom to top. The following parameters were used: $(g, \kappa, \gamma^s)/2\pi = (85, 28.3, 0.19)$ MHz.

To observe phase bistability experimentally, it is necessary to resolve the two coherent states $|\alpha_{ss}^\pm\rangle$. For strong driving this requires the ratio g/κ to be as large as possible. Also, as energy relaxation couples the two oscillators, $|n+1, \pm\rangle \xrightarrow{\gamma^s} \pm 1/\sqrt{2}(|n, +\rangle + |n, -\rangle)$, it can be shown [11] that phase bistability only occurs when $\gamma^s < 2\kappa$. As circuit-QED systems can fulfil each of these requirements, and do not suffer from problems associated with moving atoms in cavity QED [20], circuit-QED is an ideal system for the observation of phase bistability.

For large driving amplitude the steady state Q-function can be found after appropriate transformations of the density matrix [21],

$$Q(x + iy) = \frac{2e^{-\left(x - \frac{2\mathcal{E}}{\kappa}\right)^2}}{2^{\frac{\gamma^s}{\kappa}}\pi\beta\left(\frac{\gamma^s}{2\kappa}, \frac{\gamma^s}{2\kappa}\right)} \int_{-1}^{1} \frac{e^{-\left(\frac{g}{\kappa}z - y\right)^2}}{(1 - z^2)^{\left(1 - \frac{\gamma^s}{2\kappa}\right)}} dz, \quad (9)$$

where, $\beta(a, b)$ is the beta function. Simulations confirm that (9) matches the numerical solution of the master equation (1) for large $\mathcal{E}$. A cross section of this function is shown in Fig. 3 for realistic circuit-QED parameters. It is clear that $|\alpha_{ss}^{\pm}\rangle$ can be easily resolved using homodyne measurement of the field [17, 20]. The inset in Fig. 3 compares the peak locations $\alpha_{ss}^{\pm}$ to those obtained from numerical solution of (1) for a range of driving amplitudes. At small driving amplitudes the system is too anharmonic to approximate as two harmonic oscillators. However, for larger amplitudes the two solutions coincide.

When there are N qubits in the resonator an analogous phenomenon occurs: phase multistability [11]. Similarly to the single qubit case, phase multistability results from the energy structure of Tavis-Cummings manifolds at large excitation. For a given l, there are, $2l+1$, transitions between the Dicke states $|n\rangle|l,m\rangle \leftrightarrow |n+1\rangle|l,m\rangle$. The difference in energy between successive Tavis-Cummings manifolds at large intra-cavity photon number is, $E_{n+1}^{(l,m)} - E_n^{(l,m)} \approx \omega_r + m\bar{g}_N/\sqrt{n}$ [11]. Proceeding as before, assuming $\gamma_j^s \to 0$ and $g_{N,j} \approx \bar{g}_N$, strong resonant driving of the TLR leads to transitions on $2l+1$ separate ladders. The steady state density matrix of the system can be approximated by a mixture of $2l+1$ damped, uncoupled harmonic oscillators driven off resonantly. Each harmonic oscillator has the coherent state amplitude, $\alpha_{ss}^{(l,m)} = 2f_m(\mathcal{E}f_m + im\bar{g}_N)/\kappa$, where $f_m = \sqrt{1-(m\bar{g}_N/\mathcal{E})^2}$.

Due to the small ratio of $\bar{g}_N/\kappa$ and the inability to couple several qubits identically to a common field mode, phase multistability has not been experimentally demonstrated. However, as the circuit-QED based system considered here can satisfy the aforementioned parameter requirements, exploration of phase multistability is possible. Figure 4 shows the Q-function obtained by numerical solution of (1) with $H_{\mathcal{E}}$ for three qubits. The peak positions $\alpha_{ss}^{(l,m)}$ coincide with the numerical solution to high precision. The center two peaks are larger because the $m = 1/2$ transitions are driven closer to resonance than the $m = 3/2$ transitions. Phase multistability may have similar applications as phase bistability, i.e in quantum feedback [22] and nanophotonic switching [20]. Moreover, it may be possible to use phase multistability to perform a collective measurement of the system to determine the collective spin, m [22].

In summary, we have demonstrated that two fundamental collective quantum optical phenomena, superradiance and phase multistability, can be observed in a circuit-QED based system using existing experiments. This in turn allows further study of the collective phenomena. We note that these phenomena should also be observable in other types of superconducting qubits.

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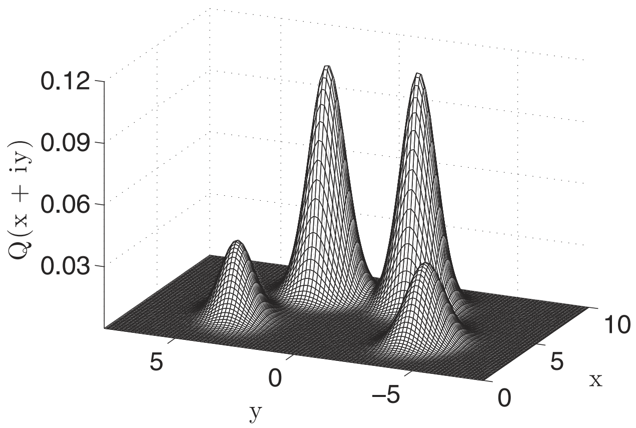

FIG. 4: Steady state Q-function for the field of a TLR with three qubits obtained from numerical solution of equations (1) with $H_{\mathcal{E}}$. Center peaks correspond to $\alpha_{ss}^{(3/2,\pm1/2)}$, whilst the other peaks correspond to $\alpha_{ss}^{(3/2,\pm3/2)}$. Parameters used are $(\mathcal{E},\kappa,\gamma_j^s)/2\pi = (169.6, 42.4, 0.19)$ MHz and $g_{3,j}/2\pi = (83.7, 85.7, 85.1)$ MHz [10]. Note that the distance between the peaks can be increased by choosing smaller κ.

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