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Quantum computational webs

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We discuss the notion of quantum computational webs: These are quantum states universal for measurement-based computation, which can be built up from a collection of simple primitives. The primitive elements—reminiscent of building blocks in a construction kit—are (i) one-dimensional states (computational quantum wires) with the power to process one logical qubit and (ii) suitable couplings, which connect the wires to a computationally universal web. All elements are preparable by nearest-neighbor interactions in a single pass, of the kind accessible in a number of physical architectures. We provide a complete classification of qubit wires, a physically well-motivated class of universal resources that can be fully understood. Finally, we sketch possible realizations in superlattices and explore the power of coupling mechanisms based on Ising or exchange interactions.

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It is an intriguing fact that one can perform universal quantum computation just by performing local measurements on certain quantum many-body systems [1–7]. Despite enormous interest in this phenomenon, our understanding of which quantum systems offer a quantum computational speedup and which do not is still rudimentary. Indeed, for years, the only states known to be universal for quantum computation by measurements were the cluster state and very close relatives [1,2,8]. This was unsatisfactory both from a fundamental point of view and for experimentalists who aimed to tailor resource states to their physical systems in the laboratory. In Refs. [6,7], a framework for the construction of new schemes for measurement-based quantum computation (MBQC) was introduced [further applied, for example, in Refs. [9,10]]. There, it was shown that many of the singular properties of the cluster are *not* necessary for a computational speedup, thus weakening the requirements for MBQC. This newly found flexibility notwithstanding, it has been established that universality is a rare property among quantum many-body states [11]. Therefore, it would be very desirable to obtain a full classification of the relatively few states that are universal. While the unqualified problem still seems daunting, in this Rapid Communication, we show that under reasonable physically motivated constraints, a complete understanding is possible.

The basic idea is to break up resource states into smaller primitives, which are more amenable to analysis. Indeed, most known states universal for MBQC come in two versions: (i) states on a one-dimensional (1D) chain of qubits, which have the ability to transport and to process one logical qubit worth of quantum information [1,6,7,9,10], and (ii) two-dimensional (2D) versions, obtained by suitably entangling several 1D strands. We will refer to such 1D states as *quantum computational wires*. They form the measurement-based equivalent of a single qubit. Likewise, the *couplings* used to form truly universal 2D resources (referred to as *quantum computational webs*) are the analogs of entangling unitaries in the gate model. To split the analysis of universal states into wires and couplings has two advantages: (i) The primitives are far easier to understand than the compound state they give rise to, and (ii) in a manner reminiscent of a construction kit, wires and couplings may be freely combined to form diverse sets of universal resources (cf. Fig. 1).

Full classification of qubit wires. For most of what follows, we focus on qubit systems for which we can provide a full theory. This constitutes our main technical result. We impose the physically reasonable requirement that wires can be built up from product states by means of nearest-neighbor interactions $U = e^{-itH^{(i,i+1)}}$ in a single translationally invariant pass. Here, the physical realizations we have in mind are atoms in an optical lattice as in an atomic sorting device [12], settings that exploit optical superlattices [9,13], or other architectures, such as ones that involve interacting quantum dots [14] or instances of networks [15]. More specifically, by a *qubit computational wire* we mean

(i) a family of pure states $|\phi_n\rangle$ of an n-qubit spin chain,

(ii) preparable from a product state $|0,\ldots,0\rangle$ by the sequential action of a unitary gate U:

$$|\psi_n\rangle = U^{(n,n-1)} \cdots U^{(3,2)} U^{(2,1)} |0,\ldots,0\rangle. \quad (1)$$

(iii) In the limit of large n, the entanglement between the left and the right half of the chain (in the sense of an area law) approaches one ebit.

These axioms may seem surprisingly weak: Earlier, we loosely characterized computational wires as states with the power to transport and to process one logical qubit. It is one central result of this Rapid Communication that any state that fulfills (i)–(iii) is automatically useful for information processing. In the following, we will prove the following complete classification of qubit wires up to local-basis changes:

Observation 1 (Classification of qubit wires). There is a three-parameter family of computational qubit wires. A wire is specified by an

(a) always-on operation $W \in SU(2)$, which acts on correlation space (see the following) after every step, independent of the basis chosen or the measurement outcome, and a

(b) by-product angle ϕ, which specifies how sensitive the resource is to the inherent randomness of measurements.

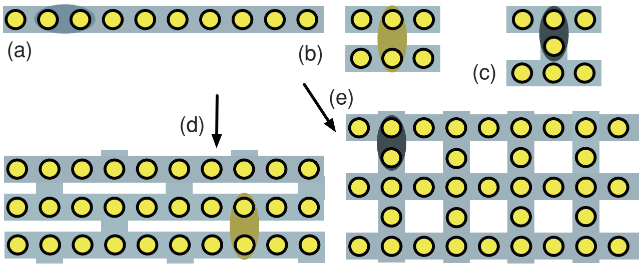

FIG. 1. (Color online) Sketch of the primitives from which one can built up new models for computing: (a) A general quantum computational wire. Two different coupling schemes based on (c), (e) an Ising-type interaction or (b), (d) Heisenberg-type or exchange interaction (the latter is defined for cluster wires).

To make sense of this statement, first note that any $|\psi_n\rangle$ has a *matrix product state* (MPS) representation [6,7,16]:

$$|\psi_n\rangle = \sum_{x_1,\ldots,x_n} \langle x_n|A[x_{n-1}]\cdots A[x_1]|0\rangle|x_1,\ldots,x_n\rangle, \qquad (2)$$

where $x_i \in \{0,1\}$ and $A[0],A[1]$ are 2×2 matrices. [Equation (2) follows from Eq. (1) by setting $A[x]_{i,j} = \langle i,x|U|0,j\rangle$.) The auxiliary 2D vector space the matrices $A[0/1]$ act on is called *correlation space*. We very briefly recall the basic idea of Refs. [6,7]. Let $|\phi^{(i)}\rangle = c_0^{(i)}|0\rangle + c_1^{(i)}|1\rangle$ be a local-state vector, and set $A[\phi^{(i)}] = \bar{c}_0^{(i)}A[0] + \bar{c}_1^{(i)}A[1]$. Then,

$$(\langle\phi^{(1)}|\otimes\cdots\otimes\langle\phi^{(n)}|)|\psi_n\rangle = \langle\phi_n|A[\phi_{n-1}]\cdots A[\phi_1]|0\rangle.$$

Hence, a local measurement with an outcome that corresponds to $|\phi_i\rangle$ is connected with the action of the operator $A[\phi_i]$ on the correlation space. MBQC can be understood completely in terms of this relation between local measurements and logical computations on correlation space [6,7]. With these notions, the precise statement of Observation 1 is that any wire allows for an MPS representation with matrices,

$$B[0] = 2^{-1/2}W, \qquad B[1] = 2^{-1/2}WS(\phi), \qquad (3)$$

where $S(\phi) = \mathrm{diag}(e^{-i\phi/2},e^{i\phi/2})$; see Fig. 2(a). [That is to say, any matrix that arises from Eq. (2) can be brought into this form by a suitable rescaling and conjugation; see the following.]

Observation 1 goes a long way toward understanding the structure of qubit wires. Assume that we measure site by site in the computational basis. By Eq. (3), at every step, the same *always-on operation* W will be applied to the correlation space, irrespective of the measurement outcome. Some tribute must be paid to the random nature of quantum measurements. It comes in the form of the *by-product* operation $S(\phi)$, which acts on the correlation system in case the "wrong" measurement outcome ($|1\rangle$ instead of $|0\rangle$) is obtained. It is remarkable that this penalty is described by a single parameter: the *by-product angle* ϕ.[1]

Examples of qubit wires. The paradigmatic qubit wire is the *cluster state*. Here, $W = H$ is the Hadamard gate, and $\phi = \pi$ is the highest possible value.[2] Thus, we can put two well-known properties of the cluster into a more general context: (i) In every step, a Hadamard gate H is applied to the logical qubit, and (ii) a wrong measurement outcome causes the application of an extra $S(\pi) \simeq \sigma_z$ gate on correlation space.

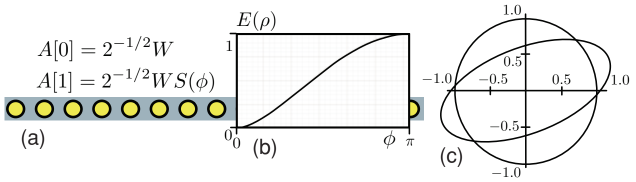

FIG. 2. (Color online) (a) Normal form of a qubit wire, (b) entropy of entanglement of a single site as a function of the by-product angle, and (c) trajectory of all operations realizable in a wire with $\phi = \pi$ (circle) and $\phi = \pi/2$ (ellipse). Every point $\sqrt{p}e^{i\delta}$ on the curve corresponds to the operation $S(-2\delta)$, realizable with probability p.

Another interesting new resource where the role of the by-product angle can be highlighted is the *T resource*, named after the common notation $T = S(\pi/2)$ for a phase gate. Here, we take $W = H$ (as for the cluster), but the by-product angle is just $\phi = \pi/2$ (so that a measurement in the computational basis gives rise to either H or HT). This qubit wire has a nonmaximal entropy of entanglement of a single site with respect to the rest of the lattice. The intuitive explanation is that T is close to the identity, so the state of the correlation system (and, hence, the rest of the chain) does not strongly depend on the outcomes of local measurements on any given site.

The proof of Observation 1 will make repeated use of the theory of MPSs [16] and of qubit channels [17]. Any MPS can be represented with matrices such that $A[0]^\dagger A[0] + A[1]^\dagger A[1] = \mathbb{1}$ [16]. The matrices give rise to a trace-preserving channel $\rho \mapsto \mathbb{E}(\rho) = \sum_x A[x]\rho A[x]^\dagger$. If one assumes that $\mathbb{E}$ has a spectral gap,[3] the half chains share one ebit of entanglement iff $\mathbb{E}$ is unital [16]. In this case, it follows easily from Ref. [17] that $\mathbb{E}(\rho) = p_0 U_0 \rho U_0^\dagger + p_1 U_1 \rho U_1^\dagger$, with suitable $U_i \in SU(2)$. From the basic theory of quantum channels, we know that there is a unitary $V \in SU(2)$ such that $p_i^{1/2}U_i = \sum_j V_{i,j}A[j]$. With that being nothing but the transformation rule for MPS representations under the local-basis change, we conclude that there is a basis in which $|\psi_n\rangle$ is represented with matrices $A'[i] = p_i^{1/2}U_i$. Next, an MPS does not change if both matrices are conjugated by the same operator X. There is an $X \in SU(2)$ such that $XU_0^\dagger U_1 X^\dagger = e^{i\alpha}S(\phi)$ for $\alpha,\phi \in \mathbb{R}$. To set $W = XU_0X^\dagger$ and $B[i] = XA'[i]X^\dagger$ implies $B[0] = p_0^{1/2}W$, $B[1] = p_1^{1/2}e^{i\alpha}WS(\phi)$. By performing the local-basis change $|1\rangle \mapsto e^{i\alpha}|1\rangle$, if necessary, we may assume that $\alpha = 0$. The fact that p_0, p_1 can be chosen to be $1/2$ will be explained later in a more general context. Conversely, any MPS with matrices as in Eq. (3) is a qubit wire. A translationally

[1]Note that for the definition of a qubit wire as such, we do not require the ability to compensate randomness of outcomes by exploiting a finite-group structure of the by-product operators.

[2]Note that our definition differs from the conventional one by the action of a local Hadamard gate on every site.

[3]Away from (and independent of) the boundaries, an MPS is completely specified by the matrices Eq. (3) iff the map $\mathbb{E}$ has a spectral gap [16]. This is true iff W is neither diagonal nor equal to σ_x. We will always implicitly assume this generic situation.

invariant preparation scheme can easily be derived by inverting the construction following Eq. (2).

Computation with qubit wires. So far we have shown that one can implement *some* unitary operation in a quantum wire (i.e., *transport* quantum information). In order to *process* it, one must have some freedom to choose which operation to apply. It will turn out—rather surprisingly—that two coincidences conspire to make any qubit quantum wire useful for that purpose. To that end, consider the one-parameter family of bases:

$$|0_\theta\rangle = \sin(\theta)|0\rangle + \cos(\theta)|1\rangle, \quad |1_\theta\rangle = \cos(\theta)|0\rangle - \sin(\theta)|1\rangle.$$

One may check directly that the operations $A[0_\theta] \propto W[\sin\theta \mathbb{1} + \cos\theta S(\phi)]$ are unitary up to scaling. The two unexpected coincidences are: (i) For any quantum wire, there is a continuous family of projections, which gives rise to unitary evolution, and (ii) the set these projections includes entire bases—so that measuring in these bases corresponds to a unitary logical computation regardless of the outcome.

Observation 2 (Unitary evolution). For any computational wire, a measurement in any basis from the one-parameter set $\{|0_\theta\rangle, |1_\theta\rangle\}$ induces a unitary evolution in correlation space.

Let us investigate the realizable unitaries. Clearly, $A[0_\theta]$ has the form $WU(\theta,\phi)$, where $U(\theta,\phi)$ is a diagonal matrix with eigenvalues $\lambda_\pm = \sin(\theta) + \cos(\theta)e^{\pm i\phi/2}$. Let $\delta = \arg(\lambda_+)$ and $p = |\lambda_+|^2$. Then, $U(\theta,\phi) = \sqrt{p}S(-2\delta)$, and basic MPS theory yields that the corresponding measurement outcome is obtained with probability p. Thus, for fixed ϕ, the set of phase gates $S(-2\delta)$, which is realizable, forms an ellipse; see Fig. 2(c),[4] in the complex plane with parametrization:

$$[\mathrm{Re}\lambda_+(\theta,\phi), \mathrm{Im}\lambda_+(\theta,\phi)]^T = \begin{pmatrix} 1 & \cos\phi/2 \\ 0 & \sin\phi/2 \end{pmatrix} \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix}.$$

Observation 3 (Phase gates). In any computational wire, an *arbitrary phase gate* $S(\delta)$ can be implemented in a single step.

If we leave the issue of randomness aside for a moment, we see that one can realize any unitary of the form $U = WS(\delta_n)WS(\delta_{n-1})\cdots WS(\delta_1)$ for some n. By invoking the assumption (see footnote 3), every $U \in SU(2)$ is of that form.

Observation 4 (Universal rotations). Except from a set of measure zero, all computational qubit wires allow for the implementation of any unitary $U \in SU(2)$ in correlation space.

Local properties. From MPS theory [16], one finds that the reduced state of a single site far away from the boundary is given by $\rho = \sum_{i,j} \mathrm{tr}(A[i]^\dagger A[j])|i\rangle\langle j|/2$. Explicitly:

$$\rho = \begin{pmatrix} 1 & \cos\phi/2 \\ \cos\phi/2 & 1 \end{pmatrix} \Big/ 2. \tag{4}$$

Interestingly, we see that the always-on operation W does not affect the local properties of the state. Hence, one can conclude [see Fig. 2(b)]:

Observation 5 (Small entanglement in wires). Computational wires with arbitrarily low local entanglement exist.

[4]We can now prove the earlier claim that, in the normal form Eq. (3), the weights of the two matrices may be chosen to be equal. That follows from the fact that there are two perpendicular vectors that intersect the ellipse at the same length $\sqrt{p}$.

Compensating randomness. In the preceding classification, we required a qubit wire to allow for transporting and processing one logical qubit. Also, we yet need to clarify how to deal with the inherent randomness of quantum measurements. If the always-on term W and the by-product operator $S(\phi)$ generate a finite group $\mathcal{B}$, there is a simple and efficient possibility to cope with randomness, introduced in Ref. [6]: Suppose we would like to implement $WS(\delta)$ but instead obtain a measurement outcome, which causes $WS(\delta')$ to be realized. Now, by measuring several consecutive sites in the computational basis, we effectively implement a random walk on the finite group $\mathcal{B}$ in correlation space. This random walk will visit any element of $\mathcal{B}$ after a finite expected number of steps. Hence, we will obtain $W^{-1} \in \mathcal{B}$ after several steps, which will yield a total evolution of $W^{-1}WS(\delta') = S(\delta')$. Then, one tries to implement $S(-\delta' + \delta)$, which is possible by Observation 4.[5] It remains to be shown how logical information in the correlation system can be prepared and can be read out. As for preparation, note that $A[2^{-1/2}(|0\rangle - e^{\phi/2}|1\rangle)] \propto |1\rangle\langle 1|$ has rank 1. Hence, if, after a local measurement, the outcome that corresponds to $2^{-1/2}(|0\rangle - e^{\phi/2}|1\rangle)$ is obtained, the correlation system will be in $|1\rangle$, irrespective of its previous state—so preparation is possible. A readout scheme can be devised along these lines.

Observation 6 (Preparation and readout). For any qubit wire, one can efficiently prepare the correlation system in a known state and read out the latter by local measurements.

Ising coupling. All wires introduced so far can be coupled to form a 2D state, universal for quantum computation. Remarkably, there are several coupling schemes, which work equally well for *all* 1D states so far introduced. Space limitations require us to describe only one and to be somewhat sketchy (however, all central points are explained; see Ref. [18] for further details). The coupling scheme, depicted in Fig. 3(a), is based on a setting where $\{1,2,3\}$ and $\{5,6,7\}$ belong to any wire, and 4 has been prepared in $2^{-1/2}(|0\rangle + |1\rangle)$. One now entangles sites $\{2,4\}$ and sites $\{4,6\}$ via Ising interactions in a suitable basis. More concretely, one performs a controlled-σ_z gate ($\mathrm{CZ}^{(2,4)}$) between site 2 and site 4 and then applies

$$W^{(6)}\mathrm{CZ}^{(4,6)}(W^{(6)})^\dagger, \quad W = \begin{pmatrix} 1 & 1 \\ e^{i\phi/2} & -e^{-i\phi/2} \end{pmatrix} 2^{-1/2},$$

between systems 4 and 6. To decouple the wires, just measure 4 in the computational basis. In the case of the $|0\rangle$ outcome, we have undone the coupling; a $|1\rangle$ outcome brings us back to the original state, up to the action of σ_z on site 2 and $W\sigma_z W^\dagger$ on 6. To perform an entangling gate, one measures 6 in the σ_z basis and 4 in the σ_x basis, by getting outcomes $x_4, z_6 \in \{0,1\}$, respectively. Let us assume that $x_4 + z_6$ is even. Choose γ, ε such that $e^{i\varepsilon/2}\sin\gamma = 1/2(1 - e^{i\phi})$, and let δ

[5]More generally, the method sketched previously may be implemented as soon as there is some basis $\{|0_\theta\rangle, |1_\theta\rangle\}$ such that $A[0_\theta], A[1_\theta]$ generate a finite group (up to scalars). It can be shown that, whenever one such basis exists, there is a one-parameter set of bases with the same property.[4] This gives rise to continuous families of wires in which randomness can be compensated by the same method.

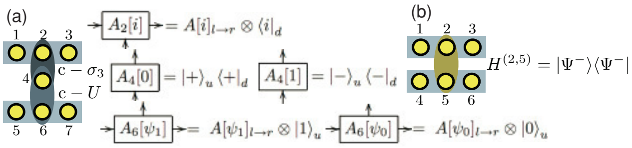

FIG. 3. (Color online) (a) Universal coupling scheme based on two (Ising-type) controlled-unitary gates for arbitrary qubit wires. For completeness, we also state the *tensor network* in the language of Refs. [6,7]. (b) A coupling of cluster wires based on an exchange interaction.

be the solution to $|\cos\delta| = |\sin\delta\sin\gamma + \cos\delta\cos\gamma e^{i\phi/2}|$ (which always exists). Finally, measure site 2 in the basis $|\psi_0\rangle = e^{-i\varepsilon}\sin\delta|0\rangle + \cos\delta|1\rangle$, $|\psi_1\rangle = -e^{-i\varepsilon}\cos\delta|0\rangle + \sin\delta|1\rangle$. A lengthy—but, by these definitions, fully specified—calculation shows that, if we get the $|\psi_0\rangle$ outcome, then one implements the unitary entangling operation,

$$\begin{aligned} V = {} & W|0\rangle\langle 0| \otimes \{\cos(\delta)A[1]\} + W|1\rangle\langle 1| \\ & \otimes\{\sin(\delta)\sin(\gamma)A[0] + \cos(\delta)\cos(\gamma)A[1]\}, \end{aligned} \quad (5)$$

between the upper and lower correlation spaces. The orthogonal outcome and the case of odd $x_4 + z_6$ may be treated analogously.

Observation 7 (Ising-type coupling). Arbitrary qubit wires can be coupled with suitable phase gates.

We use the remaining paragraphs to give an outlook on further results and ideas.

Exchange interaction coupling. By using the ideas presented earlier, one may check that cluster wires can be coupled together by using an exchange interaction Hamiltonian: $H_{\text{ex}} = |\Psi^-\rangle\langle\Psi^-|$, where $|\Psi^-\rangle = 2^{-1/2}(|0,1\rangle - |1,0\rangle)$. The topology used here is a hexagonal lattice with additional spacings; see Fig. 1(b). The coupling operation used to obtain a universal resource is given by $U = e^{i\pi/2H_{\text{ex}}}$ [18].

Observation 8 (Exchange interaction coupling). An exchange interaction Hamiltonian can be used to couple cluster wires.

Bose-Hubbard-type and continuous-variable wires. Widening our scope beyond qubits, we look at bosons in optical superlattices [9,13], subject to a *Bose-Hubbard* interaction (compare also Ref. [19]). Consider the situation where the potential forms a string of double wells, with the right site of each double well occupied by a single particle $|\Psi(t=0)\rangle = |0,1,\ldots,0,1\rangle$. In the first step, one lets the two sites of each double well interact with $H = a_L^\dagger a_R + a_R^\dagger a_L$ for time $t = \pi/4$, which leads to pairs in the state $|0,1\rangle + i|1,0\rangle$. Second—in the fashion of a quantum *cellular automaton*—one shifts the superlattice so that neighboring pairs that have not previously interacted are subjected to the preceding Hamiltonian. One obtains a globally entangled state with up to three excitations per site and entropy of entanglement between half chains of up to $E(\rho) = 1.725$. If we assume the power to perform tilted measurements in the particle number basis (or by making use of suitable internal degrees of freedom), it easily is checked that this Bose-Hubbard wire allows for the transport of one logical qubit and arbitrary rotations along one axis. This is an example of a primitive where the local Hilbert space dimension is, in principle, infinite. Further steps toward continuous-variable (CV) schemes could be done by considering correlation spaces where only a subspace of superpositions of finitely many coherent states is occupied such that the correlation space is still finite dimensional. The framework established here forms a starting point to study such CV computational schemes.

Observation 9 (Bose-Hubbard wires). Suitable states preparable by Bose-Hubbard interactions in superlattices allow for the transport of one logical qubit.

Summary. We have introduced a toolbox of primitives for constructing new quantum computational schemes. For the qubit case, we provide a full classification. The results constitute a further step toward the goal of understanding what is ultimately needed for quantum computation and what degree of freedom there is in designing computational schemes.

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