Quantum metrology with entangled coherent states

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We present an improved phase estimation scheme employing entangled coherent states and demonstrate that the states give the smallest variance in the phase parameter in comparison to NOON, BAT and "optimal" states under perfect and lossy conditions. As these advantages emerge for very modest particle numbers, the optical version of entangled coherent state metrology is achievable with current technology.

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As full quantum computing based on very large quantum resources remains on the technological horizon for now, there is significant current interest in quantum technologies that offer genuine quantum advantage with much more modest quantum resources. Quantum metrology is one field where such technologies can emerge. Non-classical states of light can offer enhanced imaging or spatial resolution, non-classical states of mechanical systems could offer enhanced displacement resolution, non-classical states of spins could enable enhanced field resolution and entangled atoms could provide the ultimate accuracy for clocks. Since it became known that optical quantum states can beat the classical diffraction/shot-noise limit [1], in recent years quantum metrology has been widely investigated in partnership with the rapid-developing field of quantum information [2]. For example, the precision limits of quantum phase measurements are given by the Cramer-Rao lower limit bounded by quantum Fisher information [3]. In the ideal quantum information version of metrology, a maximally entangled state is viewed as the best resource for quantum metrology, i.e. the optimal phase uncertainty of the NOON state reaches to the Heisenberg limit and it is thus considered for many applications (e.g. Bell's inequality tests, quantum communication and quantum computing) [4]. However, current quantum technologies have a long way to go to the manipulation of many-qubit entanglement for these applications and of course all realistic quantum technologies will be subject to loss and decoherence. Therefore, quantum metrology utilising very modest entangled resources and with robustness against loss could be accessible for these applications in the much nearer future [5], revealing a fundamental difference between classical and quantum physics in both theory and practice.

A major research question in quantum metrology is how to implement entangled NOON states with large particle numbers (called high NOON states). Many successful demonstrations have shown the potential for quantum-enhanced metrology using small NOON states [6, 7]. However, it remains a challenge to obtain a practical high NOON state in linear (or even non-linear) optics. Even if high NOON states become achievable, a critical consideration is that these states are extremely fragile to particle loss because the resultant mixed state loses phase information rapidly. Thus other quantum states have been studied for improved robustness against particle loss [8, 9]. Further recent developments have shown the potential advantages of non-linearities [10] and the importance of the query complexity for quantum metrology [11], concluding that the same phase operation is required for the appropriate resource count in different states.

In this Letter, we report that the superposition of macroscopic coherent states have an improved sensitivity of phase estimation when compared to that for NOON states, in the region of very modest photon or particle numbers. Taking into account the same *average* particle number, the entangled coherent state (ECS) overcomes the Heisenberg limit provided by a NOON state in the case of no particle loss, with this advantaged maintained over other quantum states (e.g., NOON, BAT [9], and uncorrelated states) in the case of particle loss. So even though simple coherent states $|\alpha\rangle$ are known as the most "classical-like" quantum states [12], superpositions thereof are very useful and robust for quantum metrology [13]. This phenomenon can be understood as follows. For pure states, the ECS can be understood as a superposition of NOON states with different photon numbers, thus the larger photon-number NOON states make a contribution to a better sensitivity than the average photon-number NOON state in the ECS. For mixed states, the resultant state given by photon loss does not depend on the number of particles lost but its loss rate—thus this state still contains some phase information even in the large loss rate. In order to demonstrate this perhaps surprising phenomenon, we suggest an implementation scheme using parity measurement for relatively small ECSs which are feasible with current optical technology [14].

We choose to compare the phase uncertainty of various quantum states with and without loss, using the widely- accepted approach of quantum Fisher information [3]. The interferometric set-up generally consists of four steps. The first is the preparation step where the input state $|\psi_K^{in}\rangle_{12}$ is prepared in modes 1 and 2. Then, a unitary operation U in mode 2 is applied, given by

$$U(\phi, k) = \mathrm{e}^{i\phi(a_2^\dagger a_2)^k} \tag{1}$$

for phase ϕ, order parameter of non-linearity k, and creation operator $a_i^\dagger$ in mode i. In this Letter we assume $k = 1$ implying that the operation $U(\phi, 1)$ is a conventional phase shifter $U(\phi)$ (although future studies will extend this to other k values). The outcome state is called $|\psi_K^{out}\rangle_{12} = (\mathbb{1} \otimes U(\phi))|\psi_K^{in}\rangle_{12}$. For the case of particle loss, we add two variable beam-splitters (BSs) with loss modes 3 and 4 located after the phase operation. After the BSs, the mixed state ρ_{12}^K (given by tracing out the loss modes 3 and 4) is finally measured for the estimation of phase uncertainty. A change of transmission rate T in the BSs characterises the robustness of phase estimation for the input state against the loss. The phase optimization given by the quantum Cramér-Rao bound [3] for the outcome states $|\psi_K^{out}\rangle$ is described by

$$\delta\phi_K \geq \frac{1}{\sqrt{F_K^Q}}. \tag{2}$$

For a pure state, quantum Fisher information is given by

$$F_K^Q = 4\left[\langle\psi'_K|\psi'_K\rangle - |\langle\psi'_K|\psi_K^{out}\rangle|^2\right] \tag{3}$$

for $|\psi'_K\rangle = \partial|\psi_K^{out}\rangle/\partial\phi$ [9, 15]. If the outcome state is the mixed state ρ_{12}^K, the quantum Fisher information is given by

$$F_K^Q = \sum_{i,j} \frac{2}{\lambda_i + \lambda_j} |\langle\lambda_i|(\partial\rho_{12}^K(\phi)/\partial\phi)|\lambda_j\rangle|^2, \tag{4}$$

where λ_i ($|\lambda_i\rangle$) are the eigenvalues (eigenvectors) of ρ_{12}^K.

Here we focus on three important input states as $|\psi_K^{in}\rangle$ ($K = N, B, C$) corresponding to NOON $|\psi_N^{in}\rangle$, BAT $|\psi_B^{in}\rangle$ [16], and ECS [17] given by

$$\begin{aligned} |\psi_{C_\alpha}^{in}\rangle_{12} &= \mathrm{e}^{-\frac{|\alpha|^2}{2}} \mathcal{N}_\alpha \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} \left[(a_1^\dagger)^n + (a_2^\dagger)^n\right] |0\rangle_1|0\rangle_2, \\ &= \mathcal{N}_\alpha\left[|\alpha\rangle_1|0\rangle_2 + |0\rangle_1|\alpha\rangle_2\right], \end{aligned} \tag{5}$$

where $|0\rangle_i$ and $|\alpha\rangle_i$ are respectively Fock vacuum and coherent states in spatial mode i and ($\mathcal{N}_\alpha = 1/\sqrt{2(1 + \mathrm{e}^{-|\alpha|^2})}$) [12]. Note that $|\psi_C^{in}\rangle$ can be understood as a superposition of NOON states [17] and the phase operation is imprinted in the outcome state given by

$$|\psi_{C_\alpha}^{out}\rangle_{12} = \mathcal{N}_\alpha\left[|\alpha\rangle_1|0\rangle_2 + |0\rangle_1|\alpha \mathrm{e}^{i\phi}\rangle_2\right]. \tag{6}$$

Considering first the situation with no loss, the optimal phase estimation of the pure states is analytically achievable. For the NOON and BAT states, it is equal to $\delta\phi_N \geq 1/N$ and $\delta\phi_B \geq 1/\sqrt{N(N/2+1)}$, respectively, and for the ECS

$$\delta\phi_C \geq \frac{1}{2\alpha\mathcal{N}_\alpha\sqrt{(\alpha^2+1) - (\mathcal{N}_\alpha)^2\alpha^2}}. \tag{7}$$

Taking into account equivalent resource counts for the states [18], we consider the same average photon number for mode 1 given by

$$\langle n_K\rangle = \langle\psi_K^{in}|a_1^\dagger a_1|\psi_K^{in}\rangle = \frac{N}{2} = \mathcal{N}_\alpha^2 \cdot |\alpha|^2. \tag{8}$$

Then, the phase uncertainty for the ECS can be compared with respect to N for the NOON and BAT states as shown in Fig. 2. When N becomes large, $\delta\phi_C \approx \delta\phi_N$ which indicates that the ECS becomes approximately equivalent to the NOON state, being dominated by the NOON amplitude at $N = |\alpha|^2$. However, interestingly, $\delta\phi_N$ is significantly bigger than $\delta\phi_C$ for small N because $|\psi_C^{in}\rangle$ contains a superposition of NOON states including N values exceeding $|\alpha|^2$. Furthermore, for small α, the two terms in Eq. (5) are not orthogonal (and only tend to being so in the large α limit). These superposition properties enable an advantage for the coherent states at small $|\alpha|^2$. For a more detailed example, taking $N = 4$ for the NOON and BAT states $\langle n_{N_4}\rangle = \langle n_{B_4}\rangle = 2$ and $\alpha = 2.0$ for the ECS (which gives a slightly lower resource count $\langle n_{C_2}\rangle = 1.964$), the values of the optimal phase estimation are equal to $\delta\phi_{N_4} = 0.25$, $\delta\phi_{B_4} \approx 0.289$, and $\delta\phi_{C_2} \approx 0.205$. This indicates that even with a slight resource disadvantage $\langle n_{C_2}\rangle < \langle n_{N_4}\rangle = \langle n_{B_4}\rangle$, there is still

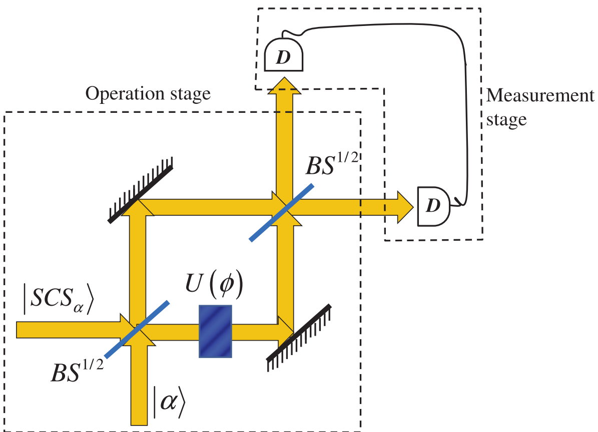

FIG. 1: It shows an interferometric setup for the ECS. Two input states ($|SCS_\alpha\rangle$ and $|\alpha\rangle$ are applied to the first BS and become the ECS. After a phase shifter $U(\phi)$ in a mode, the parity measurement is performed at the measurement stage.

a phase estimation advantage $\delta\phi_{C_2} < \delta\phi_{N_4} < \delta\phi_{B_4}$ (see Fig. 2 around at $N = 4$). This is all very well in the zero loss regime; however, the more important question is on the robustness of the phase uncertainty advantage in the realistic scenario of particle loss. In order to obtain quantum Fisher information for a mixed state due to particle loss, calculation eigenvalues and eigenvectors is required.

From the previous works [9], the optimal phase estimations for $\rho_{12}^{N_4}$ and $\rho_{12}^{B_4}$ are already known (see Fig. 3). Thus, we only need to focus on obtaining the phase estimation of the ECS $|\psi_{C_2}^{out}\rangle$. After two BSs with transmission rate T, the total state is written by $|\Psi_{C_2}\rangle_{1234} = BS_{1,3}^T BS_{2,4}^T |\psi_{C_2}^{out}\rangle|0\rangle_3|0\rangle_4$. Tracing out modes 3 and 4 we obtain the mixed state $\rho_{12}^{C_2} = \sum_{n,m=0}^{\infty} P_{nm}\,\rho_{nm}$ where $P_{nm} = {}_{1234}\langle\Psi_{C_2}|nm\rangle_{34}\langle nm|\Psi_{C_2}\rangle_{1234}$ is a probability of detecting photons n in mode 3 and m in mode 4. Because any number of particle loss in mode 3 (4) makes a collapsed state $|2\sqrt{T}\rangle_1$ ($|2\sqrt{T}e^{i\phi}\rangle_2$) in a single mode (at least either $n = 0$ or $m = 0$), the mixed state can be written in only two components given by

$$\rho_{12}^{C_2} = P_{00}\,\rho_{00} + \left(\sum_{n=1}^{\infty} P_{0n}\right)\rho_L, \tag{9}$$

where

$$P_{00} \;=\; \frac{e^{4T}+1}{e^4+1}, \tag{10}$$

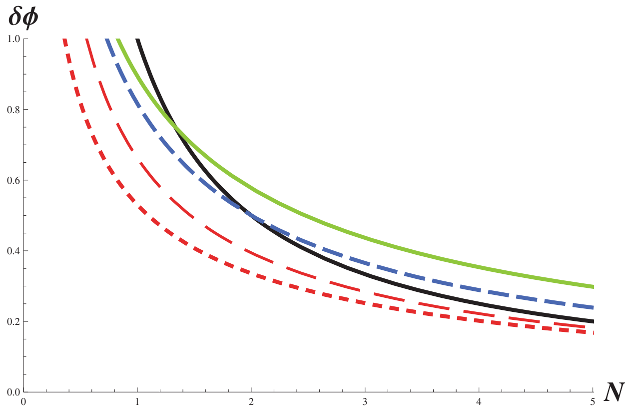

FIG. 2: The optimal phase estimations for NOON, BAT, and ECSs with no particle loss are depicted in black solid and blue dashed and red dotted lines ($\langle n\rangle = N/2 = \mathcal{N}_\alpha^2 \cdot |\alpha|^2$). Curves for NOON and BAT states are shown as continuous for comparison, but are clearly only defined at the appropriate integers N according to Eq. (8). For small N, $\delta\phi_N$ is significantly bigger than $\delta\phi_C$ while $\delta\phi_N \approx \delta\phi_C$ for large N. The cross point between δ_N and δ_B at $N = 2$ ($\alpha \approx 1.489$) indicates that the NOON and BAT states are identical. The green and red long dashed lines show the optimal phase estimation of the state given by Eq. (6) in Ref. [19] and Eq. (14), respectively.

$$P_{0n} \;=\; P_{n0} = \frac{4^n(1-T)^n}{n!}\,(\mathcal{N}_2)^2\,e^{4T-4}, \tag{11}$$

$\rho_{00} = |S_{00}\rangle_{12}\langle S_{00}|$ and

$$\rho_L = \rho_{n0} + \rho_{0n} = |S_{n0}\rangle_{12}\langle S_{n0}| + |S_{0n}\rangle_{12}\langle S_{0n}| \tag{12}$$

given by $|S_{00}\rangle_{12} = \mathcal{N}_{2\sqrt{T}}\left(|2\sqrt{T}\rangle_1|0\rangle_2 + |0\rangle_1|2\sqrt{T}e^{i\phi}\rangle_2\right)$, $|S_{n0}\rangle_{12} = |2\sqrt{T}\rangle_1|0\rangle_2$, and $|S_{0n}\rangle_{12} = |0\rangle_1|2\sqrt{T}\,e^{i\phi}\rangle_2$. Note that this is a mixture of a small ECS $|S_{00}\rangle_{12}$ (for no loss) and another mixed state ρ_L (for particle loss), which is a function of T but (in the end) not a function of particle loss n.

To calculate quantum Fisher information for the mixed state $\rho_{12}^{C_2}$, we truncate at $n = 15$, which gives a maximum error of approximately 10^{-5}. The mixed state in Eq. (9) is then approximately equal to $\tilde{\rho}_{12}^{C_2} = P_{00}\,\tilde{\rho}_{00} + \left(\sum_{n=1}^{15} P_{0n}\right)\tilde{\rho}_L$ where $\tilde{\rho}_{ij} = |\tilde{S}_{ij}\rangle_{12}\langle\tilde{S}_{ij}|$, $\tilde{\rho}_L = \tilde{\rho}_{n0} + \tilde{\rho}_{0n}$. Using eigenvalues and eigenvectors of the truncated density matrix $\tilde{\rho}_{12}^{C_2}$, we obtain the optimal phase estimation of $\tilde{\rho}_{12}^{C_2}$ depicted in Fig. 3. The optimal phase estimation of the entangled coherent state clearly improves on that of NOON, BAT, and uncorrelated states under conditions of loss, for essentailly the whole range of T. For $T \approx 1$, the value of the entangled coherent state follows that of the NOON state because $|\tilde{S}_{00}\rangle_{12}$ is the dominant factor of $\tilde{\rho}_{12}^{C_2}$ with large probability (see the inset in Fig. 3). However, it merges to that of the uncorrelated state at $T \ll 1$ because $\tilde{\rho}_L$ makes a major contribution in $\tilde{\rho}_{12}^{C_2}$ and is slightly better than the uncorrelated state due to phase coherence in $|\tilde{S}_{0n}\rangle_{12}$. We further remark on comparison with optimal states [15]. Due to the concavity of Fisher information, the engineering of optimal input states for a known lossy rate has been considered (so-called "optimal states") [15]. These states effectively provide a smooth

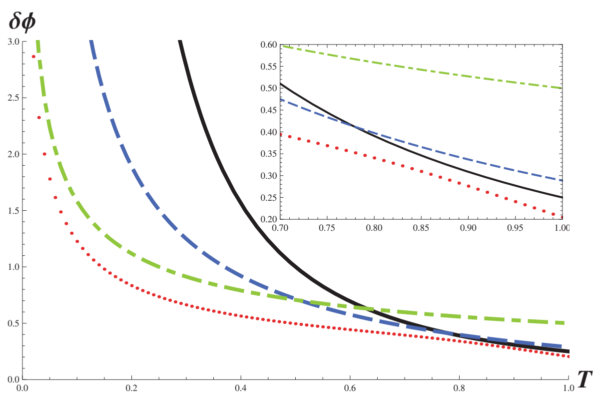

FIG. 3: The graphs show the phase uncertainty with respect to particle loss (T: transmisstion rate of the BSs) for four states ($N = 4$ and $\alpha = 2$). The black solid, blue dashed, and green dash-dotted lines indicate the NOON, BAT, and uncorrelated states. The red dotted line for the ECS shows the starting point of $\phi_{C_2} \approx 0.205$ at $T = 1$.

interpolation between NOON at high T and uncorrelated at low T, and so ECSs also offer advantage over these states.

Having demonstrated that moderate-size ECSs offer advantage with phase estimation, we also need to consider how such states could be implemented in order to realise this advantage. In principle this is achievable with current technology. There are basically four stages: 1) Generation of a Schrödinger-cat state $|SCS_\alpha\rangle = N_\alpha(|\alpha\rangle + |-\alpha\rangle)$ for $N_\alpha = 1/\sqrt{2(1+\mathrm{e}^{-2|\alpha|^2})}$. 2) When $BS^{1/2}_{1,2}$ is applied with a coherent state $|\alpha\rangle_1$ and the Schrödinger-cat state $|SCS_\alpha\rangle_2$, the resultant state becomes $|\psi^{in}_{C_{\alpha'}}\rangle = \mathcal{N}_{\alpha'}\big(|\alpha'\rangle_1|0\rangle_2 + |0\rangle_1|\alpha'\rangle_2\big)$ where $\alpha' = \sqrt{2}\alpha$. Thus, the state $|SCS_{\sqrt{2}}\rangle$ is required for the input state $|\psi^{in}_{C_2}\rangle$. According to experimental reports [14], $|SCS_\alpha\rangle$ with $\alpha \approx 1.5$ are already feasible in optics. 3) A typical phase shifter in mode 2 is applied and the outcome state is equal to $|\psi^{out}_{C_{\alpha'}}\rangle$. 4) Parity measurement is finally applied in mode 2 [20]. The measurement is given by the expectation value

$$\langle\Pi_2\rangle = \frac{2 + \mathrm{e}^{-|\alpha|^2\cos\phi}\left(\mathrm{e}^{-i|\alpha|^2\sin\phi} + \mathrm{e}^{i|\alpha|^2\sin\phi}\right)}{2+2\mathrm{e}^{|\alpha|^2}} \tag{13}$$

for $\Pi_2 = \mathrm{e}^{i\pi b_2^\dagger b_2}$. Thus, the phase variance provided by the performance of parity measurement is given by

$$\Delta\phi_{PM} = \left[\frac{1-\langle\Pi_2\rangle^2}{(\partial\langle\Pi_2\rangle/\partial\phi)^2}\right]^{\frac{1}{2}}. \tag{14}$$

As shown in the red long dashed line in Fig. 2, the parity measurement on the ECS does not saturate the optimal phase estimation given by the quantum Fisher information for this state. However, it still beats the Heisenberg limit provided by the NOON state—and with an experimentally feasible final measurement stage.

In summary, we have evaluated analytically and numerically the phase uncertainty of the ECS and showed that the state can beat the Heisenberg limit given by NOON and other states possessing the same mean particle number, for the realistic scenarios of small particle number and loss. In current optical technology, it is already feasible to obtain a travelling Schrödinger-cat state which is a key ingredient for the ECS. Although a final parity measurement would not saturate our derived phase uncertainty bound, such a realistic measurement approach could still demonstrate an advantage over NOON and other states with current technology. Mixing squeezed and coherent states and non-linearity of the phase operation [10] have been recently studied [21]. Study of the effects of squeezing variables in the Schrödinger-cat state and investigation of non-linear effects in the phase operation therefore form very interesting future research avenues.

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