# Quantum metrology with imperfect states and detectors

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We study analytically the performance of twin Fock states states in quantum metrology, showing that the Heisenberg limit for phase estimation can be attained with photon number resolving detectors when there are no losses. In a realistic scenario, involving not only losses in the interferometer, but also imperfections in state preparation and detection, we show that these states deliver close to the maximal possible precision. Our analysis identifies the tradeoffs among these types of imperfections in a demonstration of performance surpassing the standard quantum limit. In particular, we find the losses in the interferometer to be the least damaging to surpassing the standard quantum limit; the worst being detector imperfections.

Keywords: Phase estimation, losses, Fisher information

Measurements can be made more precise by using sensor designs based on quantum mechanics rather that classical physical principles. The proximate cause of this enhanced precision is the reduced measurement noise enabled by quantum entanglement. The realization of these advantages therefore hinges upon the preparation of particular nonclassical states that encode the sensor state parameter in such a manner as to allow its determination with exquisite precision [1]. Given a quantum state, the ultimate limit on the attainable precision is provided by the quantum Cramer-Rao bound via the quantum Fisher information [2]. Early theoretical efforts in quantum metrology centered around designing quantum states that saturate this bound.

A paradigm for quantum enhanced measurement is optical interferometry, in which the phase difference between two field modes is to be estimated. A schematic of such a sensor is shown in Fig. (1). When the number of input photons is fixed, and there are no losses, the quantum states minimizing the QCRB are so-called $N00N$ states, consisting of a superposition of $N$ photons in one mode and none in the other [3]. A number of epxerimental works have explored the capabilities of these states. [4, 5] Unfortunately, $N00N$ states are exponentially more vulnerable to losses than classical states, and quickly lose their capacity for enhanced sensing. This motivated the search for states that are optimized to be resilient to losses [6–8]. Rudimentary experimental studies have also been undertaken with such states [9].

The optimal states for lossy phase estimation [6–8] are, not surprisingly, dependent on the exact value of the loss parameter. Consequently, no universal scheme for their preparation is possible. Additionally, this maximum precision assumes the ability to perform on the final quantum state certain optimal measurements, given by the eigenvectors of the so-called symmetric logarithmic derivative (SLD), which captures the differential changes in the state along a trajectory generated by the parameter. Such a measurement always exists [2], but is in general prohibitively complex, because not only does it involve projections onto entangled states, but also depends on the loss in the interferometer.

In this Letter, we concentrate on a different class of states for optical interferometry, proposed by Holland and Burnett [10]. The scheme, shown in Fig. (1), starts with $N$ photons in each of two modes given by $|\Psi\rangle = |N\rangle|N\rangle$, which can be generated in a heralded manner with nonlinear processes like parametric down-conversion *etc.* and photon-number-resolving detectors (PNRDs) [11], incident onto a 50:50 beam splitter. The resulting state, which we denote HB($N$), has a photon number variance quadratic in $N$, thereby capable of attaining the Heisenberg limit for phase estimation [2]. In contrast, $N00N$ states require not only the generation of $N$ photons, but also a manipulation of these photons by means of a complex linear-optical network [13]. The output of such a network is probabilistic since it relies on a particular detection (or nondetection) event of ancillary photons. This success probability usually decreases exponentially with increasing photon numbers. Schemes that can, in principle, generate $N00N$ state with high success probability require either very high nonlinearity [14] or actively controlled cavities [15], which challenge the current technology. This decreasing probability of production necessitates post-selection on the outcomes to exhibit any quantum enhancements.

Recent work has demonstrated a scalable route to prepare HB($N$) states, relying on production of Fock states without complex linear-optical networks [11]. They are more feasible in terms of laboratory resources than $N00N$ and optimal states, yet their performance in not drastically diminished in the presence of losses [12].

First, we show that for these states, the quantum Fisher information for phase estimation can be achieved with PNRDs. The Fisher information also allows for an objective, system-independent, resource based certificate for our metrology scheme. Furthermore, we analytically study the performance of these states resulting from an imperfect preparation procedure and imperfect detectors. Any practical implementation of quantum metrology will inevitably have such imperfections. Finally, we identify the range of imperfections and losses under which we can still demonstrate an objective advantage over classical phase estimation. This allows us to pinpoint exactly the tradeoffs involved and the bottlenecks lying in the path of demonstrating quantum enhanced metrology under realistic conditions.

We begin by calculating the quantum Fisher information for phase estimation attainable with HB($N$) states in an ideal interferometer (Fig. (1)). After BS1, $\sqrt{2}a^\dagger \rightarrow c^\dagger + d^\dagger, \sqrt{2}b^\dagger \rightarrow c^\dagger - d^\dagger$, and the phase shifter $c^\dagger \rightarrow e^{i\phi}c^\dagger$,

$$|\Psi\rangle = \sum_{n=0}^{N} A_n|2n, 2N{-}2n\rangle, A_n = \frac{\sqrt{2n!(2N-2n)!}}{2^N n!(N-n)!}e^{2in\phi}, \quad (1)$$

where $\phi$ is the parameter to be estimated. The quantum Fisher information quantifies the changes in the initial state as a result of its evolution characterized by a parameter, in this case the phase. This gives $d|\Psi\rangle/d\phi \equiv |\Psi_\phi\rangle = \sum_{n=0}^{N} 2nA_n|2n, 2N-2n\rangle$, as the derivative of the state with respect to $\phi$ and appears in the expression for the SLD, leading to a quantum Fisher information of [2]

$$\mathcal{J} = 4(\langle\Psi_\phi|\Psi_\phi\rangle - |\langle\Psi|\Psi_\phi\rangle|^2). \quad (2)$$

Since $\langle\Psi_\phi|\Psi_\phi\rangle = N(3N+1)/2$, and $\langle\Psi|\Psi_\phi\rangle = iN$,

$$\mathcal{J} = 2N(N+1). \quad (3)$$

This quantity, through the quantum Cramer-Rao bound, $\Delta\phi \geq 1/\sqrt{\mathcal{J}}$, provides the absolute attainable precision in phase estimation [2] using HB($N$) states. The quadratic behaviour of the quantum Fisher information with the number of particles involved shows that we attain the Heisenberg limit. The original suggestion [10] of measuring the number difference in the two modes after BS2 (Fig. (1)) contains no information about the phase to be estimated [16] but a parity measurement on one of the resulting modes provides a bound commensurate with Eq. (3). Parity measurements are possible on the field mode [18], but require additional resources including a local oscillator reference beam that is well

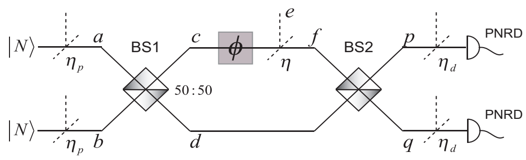


FIG. 1. A schematic interferometer involving HB($N$) states. BS1 and BS2 are 50/50 beamsplitters, and $\phi$ denotes the phase shift of mode $c$. $\eta$ is the loss in the interferometer arm, while $\eta_p$ and $\eta_d$ are the preparation and detection imperfections. $\eta = \eta_p = \eta_d = 1$ denotes a perfect setup.

matched to the probe state. Our endeavor here is to introduce a more reasonable set of measurements that attains this limit, and is more amenable to analysis in the presence of losses.

We show that having access to a beam splitter and PNRDs suffices to attain the quantum Cramer-Rao bound. Mixing modes $c$ and $d$ on BS2 yields $\sqrt{2}c^\dagger \rightarrow p^\dagger + q^\dagger, \sqrt{2}d^\dagger \rightarrow p^\dagger - q^\dagger$. Number resolving measurements $|n\rangle_p|2N-n\rangle_q$ on the two modes yields detection probabilities $p_n = \frac{n!}{(2N-n)!}\left[P_N^{N-n}(\cos\phi)\right]^2$, where $0 \leq n \leq N$, and $P_N^l(\cdot)$ are the associated Legendre polynomials. The expression for $N \leq n \leq 2N$, is obtained by substituting $n \rightarrow 2N - n$. In fact, a simple but interesting case is when we *only* make the measurement $|N\rangle|N\rangle$. The Fisher information for this situation is given by $F_N = \frac{1}{p_N(1-p_N)}\left(\frac{\partial p_N}{\partial\phi}\right)^2$, scaling exactly as the Heisenberg limit in Eq. (3). Thus, the Heisenberg limit for phase estimation with lossless interferometers can be attained with just one pair of PNRDs. This liberty is lost when the interferometer is lossy, and/or the input states and detectors imperfect. Then the required number of measurements rises quadratically with $N$, and projection only onto $|N\rangle|N\rangle$ becomes suboptimal.

*Lossy interferometry* – Analysis of the performance of HB($N$) states in interferometry in the presence of losses starts with Eq. (1), the loss in a single arm of the interferometer being modeled as $c^\dagger \rightarrow \sqrt{\eta}f^\dagger + \sqrt{1-\eta}e^\dagger$, $e$ being an unaccessible environment mode. Loss is typically allied with the phase accumulation due to a sample being measured, thus motivating treatment of loss in only one arm. Loss in both arms can be treated similarly, but requires numerical analysis and is beyond the scope of the current work. The subsequent state is $|\Psi\rangle = \frac{1}{2^N}\sum_{n=0}^{N}\sum_{m=0}^{2n} C_n B_{n,m}|2n-m\rangle_f|2N-2n\rangle_d|m\rangle_e$, where $C_n = \frac{2n!}{n!}\frac{\sqrt{(2N-2n)!}}{(N-n)!}e^{2in\phi}$, $B_{n,m} = \frac{\eta^{n-m/2}(1-\eta)^{m/2}}{\sqrt{(2n-m)!\ m!}}$, and $m$ is the number of photons lost to the environment. Since this mode is to be traced over, we can rewrite the state as

$$|\Psi\rangle = \sum_{m=0}^{2N} |\psi_m\rangle|m\rangle_e, \tag{4}$$

with $|\psi_m\rangle = \frac{1}{2^N}\sum_{k=0}^{N-\lceil\frac{m}{2}\rceil} C_{k+\lceil\frac{m}{2}\rceil} B_{k+\lceil\frac{m}{2}\rceil,m}|2k\rangle_d|2N-2k\rangle_f$, for $m$ even. For odd $m$, replace $2k \to 2k+1$ in the ket. Note that here and henceforth, we omit the explicit labeling of the modes for the sake of brevity. (See Fig. (1)) Evaluating the quantum Fisher information for phase estimation with the lossy states in Eq. (4) is simplified by their block diagonal form. Setting $|\tilde{\psi}_m\rangle = |\psi_m\rangle/\sqrt{\mathfrak{N}_m}$, with $\mathfrak{N}_m = \langle\psi_m|\psi_m\rangle$, we get $\mathcal{J} = \sum_{m=0}^{2N} \mathfrak{N}_m J(|\tilde{\psi}_m\rangle)$. Here $J$ is given by Eq. (2), and leads to

$$J(|\tilde{\psi}_m\rangle) = \frac{16}{2^{2N}\mathfrak{N}_m} \sum_{k=0}^{N-\lceil\frac{m}{2}\rceil} \left(k + \left\lceil\frac{m}{2}\right\rceil\right)^2 C^2_{k+\lceil\frac{m}{2}\rceil} \times B^2_{k+\lceil\frac{m}{2}\rceil,m}\left(1 - \frac{C^2_{k+\lceil\frac{m}{2}\rceil}B^2_{k+\lceil\frac{m}{2}\rceil,m}}{\mathfrak{N}_m}\right). \tag{5}$$

The summation can be performed in closed form, but the resulting expression in not very compact. We thus restrict our attention to some particularly interesting cases. To start with, for $N = 1$,

$$\mathcal{J}_{(N=1)} = 8\frac{\eta^2}{1+\eta^2}, \tag{6}$$

which is the same as that obtained for two-photon $N00N$ states in [7], as expected, since they are identical to HB(1) states. For higher photon numbers $N00N$ and HB($N$) states differ, and HB($N$) states are more resilient to losses than the corresponding $N00N$ states with the same number of photons. This is shown in Fig. (2) for $N = 10$, where the quantum Fisher information for HB$(10)$ exceeds the standard quantum limit for $\eta > 0.45$ and adheres closely to the optimal state.

*Imperfect preparation* – We now analyse the performance of HB($N$) states in a more realistic situation where their preparation and detection is not ideal. This is more than just with an eye towards experimental demonstration, though that provides part of the motivation. A more fundamental issue which is at stake is the gap between the principle and practice of quantum metrology.

To begin we model a scenario where the input state might not necessarily be a perfect number correlated state $|N\rangle|N\rangle$, as in Fig. (1). Independent of the physical nature of the probes, having exactly an equal number of bosons in two modes is difficult to realize experimentally. In an optical implementation, Fock states can be prepared by heralding [11, 17], with less than unit efficiency. In general, we can model this situation with

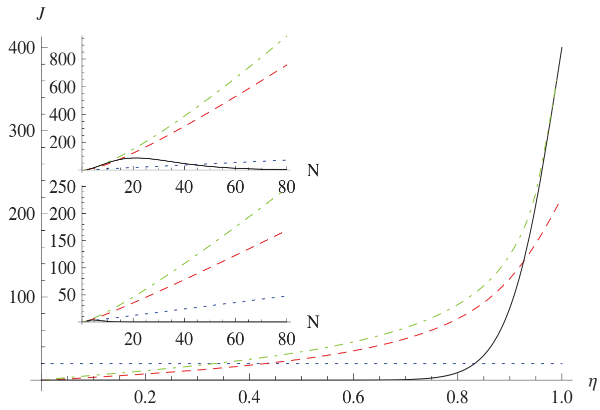


FIG. 2. Quantum Fisher information for phase estimation as a function of the transmissivity $\eta$ for 20 input photons. Blue (Dotted): Standard quantum limit, Red (Dashed): HB(10) states, Black (Solid): $N00N$ states, Green (Dot-Dashed): Optimal states [7]. Inset: Quantum Fisher information for phase estimation as a function of the photon number $N$ for $\eta = 0.9$ (top) ,and $\eta = 0.6$ (bottom).

ideal Fock state sources followed by a beam splitter of transmissivity $\eta_p$ in each mode before it is incident on the 50:50 beam splitter. Such a beam splitter leads to $|N\rangle \to \rho \equiv \sum_{n=0}^{N} \binom{N}{n} \eta_p^n(1-\eta_p)^{N-n}|n\rangle\langle n|$. The state after the 50:50 beam splitter is $U(\rho_a \otimes \rho_b)U^\dagger$, where $U = e^{i\pi(a^\dagger b + ab^\dagger)/4}$. The phase accumulation operator is given by $P = e^{i\phi a^\dagger a}$. Number resolving measurements on the two modes at the interferometer output give $p_{mn} = \langle m,n|U(P\otimes \mathbb{I}_b)U(\rho_a\otimes\rho_b)U^\dagger(P\otimes \mathbb{I}_b)^\dagger U^\dagger|m,n\rangle$. Note that the number of photons in the interferometer could now be less than $2N$. Also, $p_{mn} = 0$ if $m + n > 2N$. The resulting classical Fisher information is, in general, a function of the phase to be estimated $\phi$. The maximum is attained for $\phi = 0$, and given by $F^{\max}_{\eta_p} = 2N(N+1)\eta_p^{N+1}$. Interestingly enough, the minimum is attained for $\phi = \pi/2$, giving $F^{\min}_{\eta_p} = 2N(N+1)\eta_p^{2N}$.

*Imperfections all around* – Finally, we address the scenario where the detectors are imperfect as well. This situation is modeled, once again, by placing beam splitters in front of our number resolving detectors with transmissivity $\eta_d$. Though obtainable analytically in terms of hyper-geometric functions, the cumbersome form of the expressions for the Fisher information in the general case discourages us from presenting them here. A simple case is $F_{\eta_p,\eta_d}(\phi = 0) = 2N(N+1)(\eta_p\eta_d)^{N+1}$. Indeed, the quantum and classical Fisher information are symmetric under exchange of $\eta_p$ and $\eta_d$.

We deal with two cases, $N = 1$ and $N = 2$. These will demonstrate some subtle points involved in the estimation of the phase in a lossy interferometer involving imperfect sources and detectors, and allow us to identify regimes within which we can unambiguously demonstrate quantum advantage in metrology, once again in a lossy scenario with nonideal sources and detectors. We begin with HB(1), which has classical Fisher information $F_{(N=1)} = (8\eta_p^2\eta_d^2\eta^2(1+\eta^2)\sin^2(2\phi))/(1+\eta^4 - 2\eta^2\cos(4\phi))$. The classical Fisher information $F^{\text{perf}}_{(N=1)} \equiv F_{(N=1)}(\eta_p = \eta_d = 1)$, where the superscript perf denotes only perfect state preparation and detection, is bounded from above by Eq. (6), the quantum Fisher information, with saturation for $\phi = \pi/4$. This means that a simple adaptive technique can saturate the quantum Fisher information, even in a lossy interferometer.

To judge the performance of HB($k$) state in providing genuine quantum advantage in phase estimation, we need to surpass the corresponding standard quantum limit, given by $F^{SQL} = 2k\eta\eta_d$. This is the standard quantum limit for a classical experiment performed on an apparatus identical to the quantum one, with the assumption that the classical (coherent) state can be prepared with certainty. The figure of merit for a quantum advantage then reduces to

$$\eth_k = \frac{F_{(N=k)}}{F^{SQL}} \geq 1, \tag{7}$$

which, for HB(1) leads to

$$\eth_1 = \frac{4\eta_p^2\eta_d\eta}{1+\eta^2} > 1. \tag{8}$$

An expression like this is very beneficial, as it demonstrates the tradeoffs involved in state preparation, interferometer construction, and detection imperfection, which allows an experimentalists to direct their efforts appropriately. For instance, if $\eta_d < 0.5$, there is no way to beat the standard quantum limit with HB(1) states, thereby rendering moot any discussion about the nature of the source and the interferometer. The asymmetry between preparation and detection imperfections in the final limitations is due to the same in the standard quantum limit.

$\mathcal{J}_{(N=2)} \geq F^{\text{perf}}_{(N=2)}$, with strict inequality for some $\eta$. It means that for no phase can the classical Fisher information equal the quantum Fisher information for certain values of the the loss parameter, unlike the HB(1) case. The interesting question of the quantum advantage is again addressed by $\eth_2 = F_{(N=2)}/4\eta\eta_d$, where the right hand side is maximized over $\phi$. To get an idea of the requirements for an experiment, we find numerically that $\eth_2(0.687, 1, 1) \approx \eth_2(1, 0.135, 1) \approx \eth_2(1, 1, 0.547) \approx 1$. In general, higher photon states are more resilient to losses in the interferometer but they also put stricter demands on $\eta_p$ and $\eta_d$. Thus, with increasing photon numbers, the feasibility region would shrink along the two axes denoting the imperfections, and extend along that denoting the loss. Its also easy to see that this particular pattern is universal. The detector and preparation imperfections are identical as far as $F_{(N=k)}$ is concerned, so we can think in terms of only $\eta_p$. The attainable precision depends strongly on the input state, and $\eta_p$ affects precisely that. Thus, it is expected that $\eta_p$, and consequently $\eta_d$ has more stringent requirements placed on it that $\eta$.

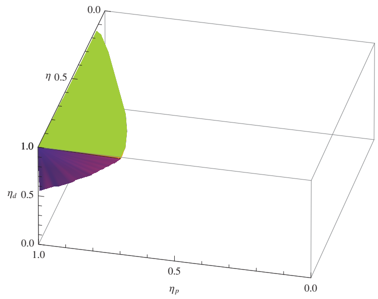


FIG. 3. Feasibility region for beating the standard quantum limit using HB(2) states. The bottleneck in beating the standard quantum limit is the detector imperfection, followed by the preparation imperfection and lastly, losses in the interferometer.

The complete region where $\eth_2(\eta_p, \eta, \eta_d) \geq 1$ is depicted in Fig. (3). To experimentally realize an improvement over its classical counterpart, quantum phase estimation with HB states requires high-quality state preparation and detection in addition to low-loss interferometers. In a realistic experiment with 95% interferometer transmission, and 60% detection efficiency (at the high end for commercially available Silicon avalanche photodiodes), the HB(1) state preparation must be better than $\eta_p \geq 0.91$, which is well beyond the current state of the art [11]. Utilizing the highest-efficiency PNRDs available, with detection efficiencies approaching 0.98 [19], relaxes the preparation of the HB(1) state to $\eta_p \geq 0.71$, which is still far from current demonstrations of $\eta_p \approx 0.245$ [11].

*Conclusions* – We have identified the tradeoffs involved in a practical demonstration of quantum enhanced metrology. This sets benchmarks for the preparation, detection, and interferometer quality. A scalable route for preparation of the HB states has been proposed, offering advantages over the $N00N$ and optimal states [11]. The optimal states are marginally better than the HB states in their attainable precision, in spite of allowing parameter dependent, nonlocal measurements at the end. It can therefore be concluded that if one considers the whole gamut of issues involved in a metrology setup, including state preparation and the final measurement, and uses the objective tool of classical and quantum Fisher information, HB states and PNRDs provide a scalable and practically realizable setup for quantum enhanced metrology.

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