Bound Nonlocality and Activation

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We investigate nonlocality distillation using measures of nonlocality based on the Elitzur-Popescu-Rohrlich decomposition. For a certain number of copies of a given nonlocal box, we define two quantities of interest: (i) the nonlocal cost and (ii) the distillable nonlocality. We find that there exist boxes whose distillable nonlocality is strictly smaller than their nonlocal cost. Thus nonlocality displays a form of irreversibility which we term "bound nonlocality." Finally, we show that nonlocal distillability can be activated.

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Entanglement and nonlocality are both powerful resources for information processing [1,2]. While entanglement has always been at the heart of quantum information science, there has recently been a growing interest in investigating nonlocality from an information-theoretic perspective. On the one hand, quantum nonlocality allows for the reduction of communication complexity [3], as well as for information processing in the device-independent setting [4], where one wants to achieve an information task without any assumption on the devices used in the protocol. On the other hand, in trying to understand why quantum nonlocality is limited [5], that is, why nonsignaling correlations stronger than those allowed in quantum mechanics do not appear to exist in nature, it has been realized that strong nonlocality enables a dramatic increase in information-theoretic power compared to the quantum case. For instance, certain postquantum correlations collapse communication complexity [6–8], violate information causality [9] and macroscopic locality [10], and outperform quantum correlations for nonlocal games [11].

In general, in order to harness the information-theoretic power offered by a given type of resource, it is essential to understand how to quantify it. While this issue is rather well developed in the case of quantum entanglement [1], much less is known for nonlocality. We lack an adequate theoretical framework for tackling this problem; thus, it is still not clear today what is a good measure of nonlocality and under which conditions two nonlocal correlations can be considered as equivalent.

The first tentative measures of nonlocality were proposed in the context of Bell experiments. One can consider, for instance, the amount of violation of a Bell inequality or the resistance to noise—or to detector inefficiency—of a given set of correlations. However, it may happen that a set of correlations does not violate a given Bell inequality even if it is nonlocal. Moreover, it was recently shown that nonlocality can be distilled [12]; that is, by locally processing several copies of certain nonlocal correlations, one can increase the amount of violation of a Bell inequality. Thus, from an information-theoretic perspective, one needs better adapted measures of nonlocality.

In the present Letter, we study measures of nonlocality based on the Elitzur-Popescu-Rohrlich (EPR2) [13] decomposition in the context of nonlocality distillation. The idea of EPR2 consists of decomposing a given correlation P into a purely local and a purely nonlocal part. The weight of the nonlocal part, minimized over all possible decompositions, then characterizes the nonlocality of P. The EPR2 decomposition provides a natural framework for studying nonlocality distillation. Given N copies of P, corresponding to the nonlocal correlation $P^{\times N}$, we identify two relevant quantities. The first one is the distillable nonlocality, which quantifies the amount of nonlocality that can be extracted from $P^{\times N}$. The second is the nonlocal cost, which quantifies how much nonlocality is required in order to build $P^{\times N}$. We investigate these two quantities for a specific class of nonlocal correlations in the two-copy scenario. Interestingly, we uncover a form of irreversibility, in that for certain undistillable correlations the nonlocal cost is strictly larger than the distillable nonlocality. We term this effect bound nonlocality in analogy to bound entanglement [14]. Exploiting a recent result of Fitzi *et al.* [15], we also provide examples of bound nonlocality for all $N > 2$, including the asymptotic limit ($N \rightarrow \infty$). Finally, we demonstrate activation of nonlocal distillability, whereby an undistillable correlation can enhance nonlocality distillation.

Measures of nonlocality.—We consider two remote parties, Alice and Bob, who share some nonlocal correlation, which from now on we shall refer to as a nonlocal box. Formally, this box is represented by a joint probability distribution $P(ab|xy)$, where x and y denote the inputs of Alice and Bob, respectively, and a and b their outputs.

We consider a measure of nonlocality based on the EPR2 decomposition [13], which consists in decomposing a nonlocal box P into a convex mixture of a local part and a nonlocal part, that is,

$$P(ab|xy) = (1 - p_{\mathrm{NL}})P_L(ab|xy) + p_{\mathrm{NL}}P_{\mathrm{NL}}(ab|xy), \quad (1)$$

where $P_L(ab|xy)$ is a local probability distribution and $P_{\mathrm{NL}}(ab|xy)$ is a no-signaling probability distribution. The nonlocality of the box P, denoted $C(P)$, is then obtained by minimizing the weight of the nonlocal part over all possible decompositions of the form (1), i.e.,

$$C(P) = \min_{\mathrm{decompositions}} p_{\mathrm{NL}}. \quad (2)$$

It follows that, for the optimal decomposition, the nonlocal part $P_{\mathrm{NL}}(ab|xy)$ has unit weight, i.e., $C(P_{\mathrm{NL}}) = 1$. $C(P)$ can be interpreted as the nonlocal cost of the box P, in the sense that this quantity represents the minimum amount of nonlocal resources required in order to construct P [16]. Geometrically, $C(P)$ can be seen as the distance between P and the set of local boxes, relative to the closest extremal nonlocal box.

An important property of $C(P)$ is that it cannot on average increase under local operations (LO) and is thus a meaningful measure of nonlocality. Here local operations include relabelling of inputs and outputs. Note that in entanglement theory, the class of operations under which entanglement cannot increase is LOCC, where CC stands for classical communication. In the case of nonlocality, communication between the parties is, however, not allowed, since it is a nonlocal resource.

A defining property of the measure $C(P)$ is that all nonlocal resources are treated on an equal footing. More precisely, all extremal nonlocal boxes are counted as equally nonlocal. As such we do not require any knowledge of the nonlocal part, namely, which extremal boxes it involves. Since characterizing extremal nonlocal boxes is a hard problem, this property appears very advantageous. Moreover, it turns out that $C(P)$ can be computed efficiently by using a linear program [15] (see below).

In the present Letter, we deal with the scenario where Alice and Bob share N copies of a given box P. Formally, they share the probability distribution

$$P^{\times N}(\mathbf{ab}|\mathbf{xy}) = P(a_1b_1|x_1y_1) \times \cdots \times P(a_Nb_N|x_Ny_N), \quad (3)$$

where we use the vector notation $\mathbf{a} = \{a_1, \ldots, a_N\}$ for the string of Alice's outputs and similarly for $\mathbf{b}$, $\mathbf{x}$, and $\mathbf{y}$.

First we would like to quantify the nonlocal cost of $P^{\times N}$. From the structure of $P^{\times N}$, it is easy to see that the weight of the nonlocal part is less than or equal to $1 - [1 - C(P)]^N$, which is simply one minus the weight of the fully local part of the N boxes. However, this represents only one possible decomposition of the form (1), and we are by no means guaranteed that it is optimal. Indeed, instead of being given N identical copies of P, we might hold a box behaving exactly as $P^{\times N}$ (i.e., represented by exactly the same probability distribution) but made by using less nonlocal resources. Therefore the correct measure of the nonlocal cost of $P^{\times N}$ is given by its EPR2 decomposition, i.e., $C(P^{\times N})$.

In the context of Clauser-Horne-Shimony-Holt (CHSH) [17], when x, y, a, and b are all bits, it was recently shown that nonlocality can be distilled [8,12]. More precisely, by LO, which now includes wiring together several copies of a box P, it is possible to obtain a box P' which contains more nonlocality. Note that in Refs. [8,12] the nonlocality of a box was measured via its CHSH value, which, in this case, coincides with the measure of nonlocality we adopt here.

It therefore appears natural to define the N-copy distillable nonlocality of a box P, to be given by the maximal nonlocality obtainable by wiring N copies of P, that is,

$$D(P^{\times N}) = \max_W C(W[P^{\times N}]), \quad (4)$$

where W is an $N \rightarrow 1$ wiring; that is, W maps $P^{\times N}$ to a box $P' = W[P^{\times N}]$ which features inputs and outputs of the same size as the initial box P. A box P is said to be N-copy distillable when $D(P^{\times N}) > C(P)$.

Now that we are in a position to quantify how much nonlocality can be extracted from N copies of a box P, it is natural to compare this quantity to $C(P^{\times N})$, the nonlocal cost of $P^{\times N}$. A first simple observation is that

$$C(P) \leq D(P^{\times N}) \leq C(P^{\times N}). \quad (5)$$

The left inequality follows from the fact that it is always possible for Alice and Bob to apply a trivial wiring, which consists in using only a single box and throwing away the $N - 1$ remaining copies. The right inequality expresses the fact that it is impossible to extract more nonlocality from a box than the amount of nonlocality actually contained in the box. Importantly, the inequalities (5) naturally link the EPR2 decomposition with nonlocality distillation: $C(P) < C(P^{\times N})$ is a necessary condition for distillation to be possible.

A natural issue to investigate is reversibility. Can the nonlocality contained in N copies of a box P always be extracted via distillation? In other words, is $D(P^{\times N}) = C(P^{\times N})$? In the following, we will answer this question in the negative. Furthermore, we will discuss an even stronger form of irreversibility. There exist boxes P which cannot be distilled, although N copies of P contain a strictly larger amount of nonlocality than a single copy. Formally, this means $C(P) = D(P^{\times N}) < C(P^{\times N})$. We term this phenomenon bound nonlocality. Below, we provide examples of bound nonlocal boxes in the two-copy setting, as well as an example in the asymptotic limit.

Bound nonlocality.—From now on we focus on the CHSH scenario, where $x, y, a, b \in \{0, 1\}$. We consider a two-dimensional section of the no-signaling polytope [18], characterized as follows:

$$P(\xi, \gamma) \equiv \xi P^{\mathrm{PR}} + \gamma P^c + (1 - \xi - \gamma)P^f \quad (6)$$

with $\xi, \gamma \geq 0$ and $\xi + \gamma \leq 1$. Here we have used the following probability distributions: $P^{\mathrm{PR}}(ab|xy) = \frac{1}{4}[1 + (-1)^{a\oplus b\oplus xy}]$ is the Popescu-Rohrlich (PR) box, where $\oplus$ is addition modulo 2; $P^c(ab|xy) = \frac{1}{4}[1 + (-1)^{a\oplus b}]$ is a local box featuring unbiased but perfectly correlated outputs; $P^f(ab|xy) = \frac{1}{8}[2 + (-1)^{a\oplus b\oplus xy}]$ is the isotropic local box sitting on the CHSH facet below the PR box (i.e., a convex mixture of the PR box and white noise). The nonlocal cost of the boxes (6) is $C[P(\xi, \gamma)] = \xi$, which follows from the fact that the PR box is the only extremal nonlocal box above the CHSH facet, on which both P^c and P^f lie. Thus (6) is the optimal EPR2 decomposition of $P(\xi, \gamma)$. Two classes of boxes that will be important below are (i) isotropic nonlocal boxes $P_{\mathrm{ISO}}(\xi) \equiv P(\xi, 0)$ and (ii) correlated nonlocal boxes $P_{\mathrm{NLC}}(\xi) \equiv P(\xi, 1-\xi)$.

The distillability of the boxes (6) was recently investigated. On the one hand, it was shown that isotropic nonlocal boxes cannot be distilled in the case of two copies [19]. On the other hand, Refs. [8,12,20,21] presented distillation protocols for correlated nonlocal boxes. Below, these protocols are referred to as follows: FWW for the protocol of Ref. [12], BS for Ref. [8], and ABLPSV for Ref. [20].

We first focus on the case $N = 2$ and characterize those boxes of the form (6) which can be distilled. It is possible to check, by an exhaustive search over all possible distillation protocols (see [8] for details), that the FWW and ABLPSV protocols are sufficient to characterize the distillable region. That is, $D[P(\xi, \gamma)^{\times 2}] > C[P(\xi, \gamma)]$ if and only if $P(\xi, \gamma)$ can be distilled via the FWW or ABLPSV protocol.

Next we compute $C[P(\xi, \gamma)^{\times 2}]$, the nonlocal cost of two copies of $P(\xi, \gamma)$. This is done via a linear program [15], which maximizes the weight of the local part, i.e., the quantity $p_L = 1 - p_{\mathrm{NL}}$. The maximal local part p_L^* is the optimal value of the linear program

$$\max \sum_{j=1}^{n} q_j, \tag{7}$$

$$\text{subject to} \quad \sum_{j=1}^{n} q_j D_j(\mathbf{ab}|\mathbf{xy}) \leq P^{\times 2}(\mathbf{ab}|\mathbf{xy}), \qquad q_j \geq 0,$$

where the first condition should be understood as a vector inequality. Here $D_j(\mathbf{ab}|\mathbf{xy})$ denote the $n = 4^8$ deterministic local strategies for the case of four inputs and four outputs. The nonlocal cost is then given by $C[P(\xi, \gamma)^{\times 2}] = 1 - p_L^*$.

The results are presented in Fig. 1. It shows three regions of qualitatively different kinds of boxes. The first region (I) corresponds to 2-distillable boxes, for which $C(P) < D(P^{\times 2})$. The second region (II) corresponds to two-copy bound nonlocal boxes, for which $C(P) = D(P^{\times 2}) < C(P^{\times 2})$. The third region (III) corresponds to boxes such that $C(P^{\times 2}) = C(P)$; we term these "HH boxes."

Indeed it would be interesting to investigate the case of more copies. Unfortunately, very little is known beyond the case $N = 2$. Still, a notable result is that of Ref. [15], where the scaling of the nonlocal part of N copies of isotropic boxes as N increases was investigated. Interestingly, they found that $C[P_{\mathrm{ISO}}(\xi)^{\times N}] > C[P_{\mathrm{ISO}}(\xi)]$ for $N \geq 3$ (see also [22]). At first sight this result seemed to suggest that isotropic boxes could be distilled by a protocol involving

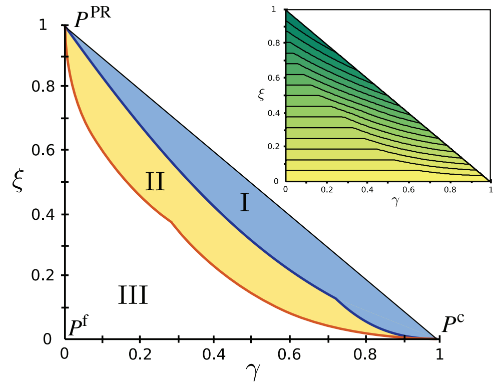

FIG. 1 (color online). Section of the no-signaling polytope given by nonlocal boxes $P(\xi, \gamma)$. The region features three qualitatively different types of boxes: (I) two-copy distillable boxes, (II) two-copy bound nonlocal boxes, and (III) HH boxes. Inset: Contour plot of the nonlocal cost of two copies, showing lines of constant $C[P^{\times 2}(\xi, \gamma)]$.

three copies or more. However, our present findings diminishes this hope. Another consequence of the result of Ref. [15] is that HH boxes exist only in the case $N = 2$. Thus this behavior is a finite size effect.

Finally, it is worth discussing the asymptotic case. A question of particular interest is whether bound nonlocality can survive in the limit $N \to \infty$. Remarkably, the answer to this question is yes. An example is the isotropic box reaching the Tsirelson bound of quantum nonlocality (CHSH $= 2\sqrt{2}$), i.e., the box $P_{\mathrm{ISO}}(\xi)$ with $\xi = \sqrt{2} - 1$. Clearly, this box cannot be distilled, even with infinitely many copies, since the set of quantum correlations is closed under wirings [20]. Nevertheless, the result of Ref. [15] shows that for $N \geq 3$ the nonlocal cost of N copies exceeds the cost of a single copy. Therefore, this box is bound nonlocal for all $N \geq 3$, including the asymptotic limit.

Activation of nonlocal distillability.—Above, we have shown the existence of restricted forms of nonlocality, such as bound nonlocality. A natural question which arises now is whether the nonlocality contained in such boxes can be activated.

Below, we show that nonlocal distillability can be activated. First, we show that for any box P_1, there exists another box P_2 such that $D(P_1 \times P_2) > \max(C(P_1), C(P_2))$. In other words, by combining one copy of P_1 and P_2, it becomes possible to achieve a task which would be impossible with one copy of either P_1 or P_2. Second, we present a stronger form of activation: For any two-copy undistillable box P_1 and integer $N \geq 1$, there exists a box P_2 such that $D(P_1 \times P_2) > \max(D(P_1^{\times 2}), D(P_2^{\times N}))$, thus showing activation of undistillable nonlocality.

Let us consider the following example. We take $P_1 = P_{\mathrm{ISO}}(\xi)$ and $P_2 = P_{\mathrm{NLC}}(\xi')$. Next we apply the BS protocol to $P_1 \times P_2$ and obtain the box $P' = W_{\mathrm{BS}}(P_1 \times P_2)$, which has nonlocal cost $C(P') = \xi + \xi'(1-\xi)/8$. For all $0 < \xi' \leq \xi < 1$, we have that $C(P') > \xi = \max(C(P_1), C(P_2))$. Thus we get activation of nonlocality.

This result holds for any box P_1 with binary inputs and outputs, since any such box can be "twirled" via LO to an isotropic box featuring the same nonlocal cost [23].

Next, we note that $D[P_{\rm NLC}^{\times N}(\xi')] \leq C[P_{\rm NLC}^{\times N}(\xi')] \leq 1-(1-\xi')^N$. Thus, in the case $0<1-(1-\xi')^N\leq \xi<1$, we obtain that $C(P') > \max(D(P_1^{\times 2}), D(P_2^{\times N}))$. In other words, by combining one copy of P_1 and P_2, we obtain a box P', with $C(P') > \xi$, which would be impossible from two copies of P_1 or from N copies of P_2.

Moreover, when P_1 reaches Tsirelson's bound, i.e., $\xi = \sqrt{2}-1$, it is actually bound nonlocal in the asymptotic limit. Remarkably, one copy of the box P_2 can activate the bound nonlocality of the box P_1 regardless of the amount of nonlocality of P_2. We note the similarity between this example and the original example of activation of bound entanglement [24]. There it was shown that by taking one copy of an entangled state ρ which cannot be distilled without collective operations, and sufficiently many copies of a bound entangled state $\sigma_{\rm BE}$, activation occurs. That is, the fidelity of ρ with a maximally entangled state can be made arbitrarily close to 1. This is indeed impossible for one copy of ρ or for arbitrarily many copies $\sigma_{\rm BE}$.

Discussion and open questions.—We investigated nonlocality distillation by using measures of nonlocality based on the EPR2 decomposition and showed the existence of bound nonlocality. In addition, we presented examples of activation of nonlocal distillability. These examples show that any nonlocal box is useful for nonlocality distillation, in the sense it can be used to boost the distillation process of other boxes.

Let us comment on some open questions. A first issue concerns the computation of the measure. While the nonlocal cost can be computed efficiently, via a linear program, we had to run an exhaustive search over distillation protocols in order to determine the distillable nonlocality. It would be interesting to find out whether distillable nonlocality can be computed more efficiently, or at least whether meaningful bounds can be derived in a simpler way. One possibility would be to get a better understanding of the structure of sets of correlations which are closed under wirings [20]. For instance, given an initial set of boxes S, how could one characterize the set of boxes which can be generated by wiring arbitrarily many copies of boxes in S? In other words, how could one find the smallest closed set containing S?

A further interesting problem concerns the asymptotic behavior of our measures. We have shown the existence of bound nonlocality for any number of copies $N \geq 2$, including the limit $N \rightarrow \infty$. There is, however, much work to be done in order to understand the asymptotic regime. In particular, while we focused here on finite copy distillation, it would be interesting to investigate asymptotic distillation.

Finally, it would be interesting to see how our measure of nonlocality relates to others. In particular, it was recently shown that the PR box can be considered as a unit of bipartite nonlocality [25], in the sense that all bipartite nonlocal boxes can be simulated arbitrarily well by using only PR boxes; note, however, that it is not known whether asymptotically PR boxes can be reversibly transformed into any nonlocal resource. Still, this suggests another natural measure of nonlocality, namely, the minimal number of PR boxes required to simulate any box. It is noteworthy that computing this measure requires detailed knowledge of the nonlocal part, which is arguably undesirable. Nevertheless, it would be interesting to understand the properties of this measure and how they relate to the measures presented here.

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