Monogamy of multipartite Bell inequality violations

P. Kurzyński,[1, 2] T. Paterek,[1] R. Ramanathan,[1] W. Laskowski,[3, 4, 5] and D. Kaszlikowski[1, 6, *]

[1] Centre for Quantum Technologies, National University of Singapore, 3 Science Drive 2, 117543 Singapore, Singapore
[2] Faculty of Physics, Adam Mickiewicz University, Umultowska 85, 61-614 Poznań, Poland
[3] Institute of Theoretical Physics and Astrophysics, University of Gdańsk, 80-952 Gdańsk, Poland
[4] Fakultät für Physik, Ludwig-Maximilians Universität München, 80755 München, Germany
[5] Max Planck Institut für Quantenoptik, 85748 Garching, Germany
[6] Department of Physics, National University of Singapore, 2 Science Drive 3, 117542 Singapore, Singapore

We show that the complementarity relation between dichotomic observables leads to the monogamy of Bell inequality violations. We introduce a simple condition for the squares of expectation values of complementary observables that is satisfied by all physical states. This condition is used to study multi-qubit correlation inequalities involving two settings per observer. In contrast with the two-qubit case a rich structure of possible violation patterns is shown to exist in the multipartite scenario.

Quantum mechanical predictions violate Bell inequalities [1]. An interesting phenomenon occurs when a system is involved in more than one Bell experiment. In this case trade-offs exist between strengths of violations of a Bell inequality by different sets of observers, known as monogamy relations [2–7]. One of the origins of this monogamy is the principle of no-signaling, according to which no information can be transmitted faster than the speed of light. If violations are sufficiently strong possibility of superluminal communication between observers arises and consequently the Bell monogamy is present in every no-signaling theory [4–7]. However, no-signaling principle alone does not identify the set of violations allowed by quantum theory. The monogamy relations derived within quantum theory, in the scenario where a Bell inequality is tested between parties AB and AC, show even more stringent constraints on the allowed violations [2, 3].

Here we derive within quantum theory the monogamy relations which involve violation of multi-partite Bell inequalities, and study their properties. The trade-offs obtained are stronger than those arising from no-signaling alone and in some cases we show that they fully characterize the quantum set of allowed Bell violations. Our method uses complementarity of operators defining quantum values of Bell parameters and shows that Bell monogamy stems from quantum complementarity.

We begin with the principle of complementarity, which forbids simultaneous knowledge of certain observables, and show that the only dichotomic complementary observables in quantum formalism are those that anti-commute. Conversely, we present a theorem that gives a bound for the sum of squared expectation values of anti-commuting operators in any physical state. This result is subsequently used to derive monogamy of Bell inequality violations. The theorem also finds other applications, for instance see Ref. [8].

Consider a set of dichotomic complementary measurements with the corresponding outcomes ± 1. The complementarity is manifested in the fact that if the expectation value of one measurement is ± 1 then expectation values of all other complementary measurements are zero. We show that the corresponding quantum mechanical operators anti-commute. Consider a pair of dichotomic operators A and B and put the expectation value $\langle A \rangle = 1$, i.e., the state being measured is one of the $+1$ eigenstates $|a\rangle$. Complementarity requires $\langle a| B |a\rangle = 0$, i.e., $B|a\rangle = |a_\perp\rangle$, a state orthogonal to $|a\rangle$. Since $B^2 = \mathbb{1}$, we also have $B|a_\perp\rangle = |a\rangle$ and therefore $|b\rangle = \frac{1}{\sqrt{2}}(|a\rangle + |a_\perp\rangle)$ is the eigenstate of B with $+1$ eigenvalue. For this state complementarity demands, $\langle b| A |b\rangle = 0$, i.e. $A|b\rangle$ is orthogonal to $|b\rangle$ which is only satisfied if $|a_\perp\rangle$ is the eigenstate of A with -1 eigenvalue. The same argument applies to all eigenstates $|a\rangle$ with $+1$ eigenvalue and therefore the two eigenspaces have equal dimension. Therefore, $A = \sum_a (|a\rangle\langle a| - |a_\perp\rangle\langle a_\perp|)$ and $B = \sum_a (|a_\perp\rangle\langle a| + |a\rangle\langle a_\perp|)$. It is now easy to verify that A and B anti-commute.

Conversely, consider a set of traceless and trace-orthogonal dichotomic hermitian operators A_k and assume for the moment that arbitrary quantum states can be decomposed as $\rho = \frac{1}{d}\left(\mathbb{1} + \sum_{k=1}^{d^2-1} \alpha_k A_k\right)$. Here, d is the dimension of the underlying Hilbert space and α_k are real coefficients in the range $[-1, 1]$ being expectation values of measurements A_k in the state ρ. Let us group operators A_k into disjoint sets S_j of mutually anti-commuting operators, $S_j = \{A_1^{(j)}, A_2^{(j)}, \dots\}$. Denoting the number of sets by $\mathcal{S}$, the density operator reads

$$\rho = \frac{1}{d}\left(\mathbb{1} + \sum_{j=1}^{\mathcal{S}} F_j\right), \tag{1}$$

where $F_j \equiv \sum_{k=1}^{|S_j|} \alpha_{kj} A_k^{(j)} = \vec{\alpha}_j \cdot \vec{A}_j$. Operators F_j are traceless, orthogonal, and have only two eigenvalues $\pm|\vec{\alpha_j}|$, because $F_j^2 = |\vec{\alpha}_j|^2 \mathbb{1}$, which follows from the properties of A_k. Moreover, the eigenvalues are equally degenerated since $\mathrm{Tr}(F_j) = 0$.

The properties of F_j operators allow derivation of the following theorem, which is a necessary condition for ρ

to describe a physical state. For a related theorem for Clifford algebra observables see Ref. [9].

Theorem: For all physical states and for all j, $|\vec{\alpha}_j| \leq 1$.

Proof: The proof is by contradiction. Let us assume the existence of an index j such that $|\vec{\alpha}_j| > 1$. We show that this implies negativity of ρ. Note that $F_j = |\vec{\alpha}_j|(\Pi_+ - \Pi_-)$, where $\Pi_\pm$ denotes a projector onto subspace of degenerated eigenvalue $\pm|\vec{\alpha}_j|$. Since $\Pi_+ + \Pi_- = \mathbb{1}$ we find

$$\Pi_- = \tfrac{1}{2}\left(\mathbb{1} - F_j/|\vec{\alpha}_j|\right). \quad (2)$$

Both eigenvalues of F_j have the same degeneracy $\mathrm{Tr}(\Pi_\pm) = d/2$, and hence the probability to observe the result associated with Π_- in the state ρ reads

$$\mathrm{Tr}(\rho\Pi_-) = \frac{1}{d}\left(\frac{d}{2} - \frac{d}{2}|\vec{\alpha}_j| - \sum_{j' \neq j} \frac{\mathrm{Tr}(F_j F_{j'})}{2|\vec{\alpha}_{j'}|}\right). \quad (3)$$

The last term vanishes due to orthogonality of operators F_j and since by assumption $|\vec{\alpha}_j| > 1$, the probability in (3) is negative, which ends the proof. $\square$

This theorem implies that if an expectation value of one observable is ± 1 then expectation values of all other anti-commuting observables are necessarily zero. In this way anti-commuting operators encode the concept of complementary observables in quantum formalism. The theorem is in fact more general as it gives trade-offs between squared expectation values of anti-commuting operators in any physical state.

We apply these results to derive constraints on Bell inequalities between many qubits. A general N-qubit density matrix can be decomposed into tensor products of Pauli operators

$$\rho = \frac{1}{2^N} \sum_{\mu_1,\ldots,\mu_N=0}^{3} T_{\mu_1\ldots\mu_N} \sigma_{\mu_1} \otimes \cdots \otimes \sigma_{\mu_N}, \quad (4)$$

where $\sigma_{\mu_n} \in \{\mathbb{1}, \sigma_x, \sigma_y, \sigma_z\}$ is the μ_n-th local Pauli operator for the n-th party and $T_{\mu_1\ldots\mu_N} = \mathrm{Tr}[\rho(\sigma_{\mu_1} \otimes \cdots \otimes \sigma_{\mu_N})]$ are the components of the correlation tensor $\hat{T}$. The orthogonal basis of tensor products of Pauli operators has the property that its elements either commute or anti-commute.

We consider a complete collection of two-setting correlation Bell inequalities for N qubits, which form a necessary and sufficient condition for existence of local hidden variable model [10–12]. This condition can be condensed into a single Bell parameter, which is bounded by one [12]. The quantum value of this parameter, denoted by $\mathcal{L}$, was shown to have an upper bound of

$$\mathcal{L}^2 \leq \sum_{k_1,\ldots,k_N=x,y} T^2_{k_1\ldots k_N}, \quad (5)$$

where summation is over orthogonal local directions x and y which span the plane of the local settings [12].

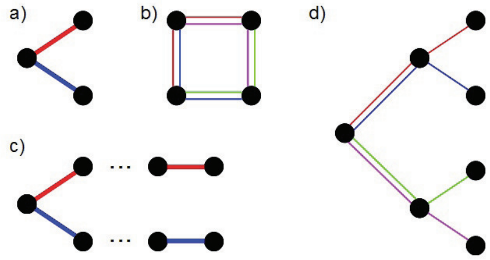

FIG. 1: The nodes of these graphs represent observers trying to violate Bell inequalities which are denoted by colored edges. **a)** The simplest case: two subsets of three parties try to violate CHSH inequality [Eq. (6)]. **b)** Four subsets of four parties try to violate Mermin inequality [Eq. (10)]. **c)** Two subsets of odd number of parties try to violate multipartite Bell inequality in a scenario in which only one particle is common to two Bell experiments [Eq. (12)]. **d)** A binary tree configuration leads to strong monogamy relation (15).

Our method for finding monogamy relations is to use condition (5) for combinations of Bell parameters and then identify sets of anti-commuting operators in order to utilize our theorem to obtain a bound on these combinations.

To describe the method, consider the simplest scenario of three particles, illustrated in Fig. 1a. There exists a monogamy relation for the violation of the CHSH inequality [13] between parties AB and AC. It was shown that for quantum systems of arbitrary dimension [3]:

$$\mathcal{B}^2_{AB} + \mathcal{B}^2_{AC} \leq 8, \quad (6)$$

where $\mathcal{B}_{AB}$ ($\mathcal{B}_{AC}$) stands for the quantum value of the CHSH parameter obtained by AB (AC). If the CHSH inequality is violated by AB then it cannot simultaneously be violated by AC. Moreover, if AB obtain maximal violation of $\mathcal{B}_{AB} = 2\sqrt{2}$, then necessarily $\mathcal{B}_{AC} = 0$, which is a form of monogamy of entanglement [4, 5, 14–17].

We now show a slightly more general result for qubits, which is the monogamy of local hidden variable correlations. We show that if correlations obtained in two-setting Bell experiment by AB cannot be modeled by local hidden variables, then correlations obtained by AC admit local hidden variable model. We use condition (5) which applied to the present bipartite scenario reads: $\mathcal{L}^2_{AB} + \mathcal{L}^2_{AC} \leq \sum_{k,l=x,y} T^2_{kl0} + \sum_{k,m=x,y} T^2_{k0m}$. It is important to note that the settings of A are the same in both sums and accordingly orthogonal local directions x and y are the same for A in both sums. We arrange the Pauli operators corresponding to correlation tensor components entering the sums into the following two sets of anti-commuting operators: $\vec{A}_1 = (XX\mathbb{1}, XY\mathbb{1}, Y\mathbb{1}X, Y\mathbb{1}Y)$ and $\vec{A}_2 = (YX\mathbb{1}, YY\mathbb{1}, X\mathbb{1}X, X\mathbb{1}Y)$, where $X = \sigma_x$ and $Y = \sigma_y$. Note that the anti-commutation of any pair of operators within a vector is solely due to anti-commutativity of local Pauli operators. From the theorem, we obtain our result

$$\mathcal{L}_{AB}^2 + \mathcal{L}_{AC}^2 \leq 2. \tag{7}$$

In particular, for qubits, Eq. (7) is more general than Eq. (6) because the CHSH inequality is a special case of the unifying inequality of Ref. [12].

Before we move to a general case of arbitrary number of qubits, we present an explicit example of multipartite monogamy relation. Consider parties A, B, C, D trying to violate a correlation Bell inequality in a scenario depicted in Fig. 1b. We shall show the following quantum bound

$$\mathcal{L}_{ABC}^2 + \mathcal{L}_{ABD}^2 + \mathcal{L}_{ACD}^2 + \mathcal{L}_{BCD}^2 \leq 4. \tag{8}$$

Condition (5) applied to these tripartite Bell parameters implies that the left-hand side is bounded by the sum of 32 elements. The corresponding tensor products of Pauli operators can be grouped into four sets

$$\begin{aligned} \vec{A}_1 &= (XXY\mathbb{1}, XY\mathbb{1}X, X\mathbb{1}XY, \mathbb{1}YYY, \dots), \\ \vec{A}_2 &= (XYX\mathbb{1}, YY\mathbb{1}Y, Y\mathbb{1}XX, \mathbb{1}XXY, \dots), \\ \vec{A}_3 &= (YXX\mathbb{1}, XX\mathbb{1}Y, Y\mathbb{1}YY, \mathbb{1}XYX, \dots), \\ \vec{A}_4 &= (YYY\mathbb{1}, YX\mathbb{1}X, X\mathbb{1}YX, \mathbb{1}YXX, \dots), \end{aligned} \tag{9}$$

where the dots denote four more operators being the previous four operators with X replaced by Y and vice versa. All operators in each vector $\vec{A}_j$ anti-commute and by virtue of our theorem, Eq. (8) is proved.

To give a concrete example of monogamy of a well-known inequality we choose the inequality due to Mermin [18]: $E_{112} + E_{121} + E_{211} - E_{222} \leq 2$, where E_{klm} denote the correlation functions. Since the classical bound of the Mermin inequality is 2, and not 1 as we have assumed in the derivation of Eq. (8), the Mermin monogamy is

$$\mathcal{M}_{ABC}^2 + \mathcal{M}_{ABD}^2 + \mathcal{M}_{ACD}^2 + \mathcal{M}_{BCD}^2 \leq 16, \tag{10}$$

where $\mathcal{M}$ is the quantum value of the corresponding Mermin parameter. The bound of the new monogamy relation (10) can be achieved in many ways. If a triple of observers share the GHZ state, they can obtain maximal violation of 4 and the remaining triples observe vanishing Mermin quantities $\mathcal{M}$. This can be attributed to maximal entanglement of the GHZ state. It is also possible for two and three triples to violate Mermin inequality non-maximally, and at the same time to achieve the bound. For example, the state $\frac{1}{2}\left(|0001\rangle + |0010\rangle + i\sqrt{2}|1111\rangle\right)$ allows ABC and ABD to obtain $\mathcal{M} = 2\sqrt{2}$, and the state $\frac{1}{\sqrt{6}}\left(|0001\rangle + |0010\rangle + |0100\rangle + i\sqrt{3}|1111\rangle\right)$ allows ABC, ABD and ACD to obtain $\mathcal{M} = \frac{4}{\sqrt{3}}$. Note that it is impossible to violate all four inequalities of (10) simultaneously.

We now derive monogamy relations for N qubits. Consider scenario of Fig. 1c, in which N is odd, A is the fixed qubit and the remaining $N-1$ qubits are split into two groups $\vec{B} = (B_1, ..., B_M)$ and $\vec{C} = (C_1, ..., C_M)$ each containing $M = \frac{1}{2}(N-1)$ qubits. We shall derive the trade-off relation between violation of $(M+1)$-partite Bell inequality by parties $A\vec{B}$ and $A\vec{C}$. Using condition (5), the elements of the correlation tensor which enter the bound of $\mathcal{L}_{A\vec{B}}^2 + \mathcal{L}_{A\vec{C}}^2$ are of the form $T_{kl_1...l_M 0...0}$ and $T_{k0...0m_1...m_M}$. The corresponding Pauli operators can be arranged into 2^M sets of four mutually anti-commuting operators each:

$$\begin{aligned} \vec{A}_{1S} &= (XXSI,\ XYSI,\ YIXS,\ YIYS), \\ \vec{A}_{2S} &= (YXSI,\ YYSI,\ XIXS,\ XIYS), \end{aligned} \tag{11}$$

where S stands for all 2^{M-1} combinations of X's and Y's for $M-1$ parties, and $I = \mathbb{1}^{\otimes M}$ is identity operator on M neighboring qubits. Therefore, according to the theorem, we arrive at the following trade-off

$$\mathcal{L}_{A\vec{B}}^2 + \mathcal{L}_{A\vec{C}}^2 \leq 2^M. \tag{12}$$

The bound of this inequality is tight in the sense that there exist quantum states achieving the bound for all allowed values of $\mathcal{L}_{A\vec{B}}$ and $\mathcal{L}_{A\vec{C}}$. This is a generalization of a similar property for CHSH monogamy [3]. The state of interest is

$$|\psi\rangle = \tfrac{1}{\sqrt{2}}\cos\alpha\left(|0\vec{0}\vec{0}\rangle + |1\vec{0}\vec{1}\rangle\right) + \tfrac{1}{\sqrt{2}}\sin\alpha\left(|1\vec{1}\vec{0}\rangle + |0\vec{1}\vec{1}\rangle\right), \tag{13}$$

where e.g. $|1\vec{0}\vec{1}\rangle$ denotes a state in which qubit A is in the $|1\rangle$ eigenstate of local Z basis, all qubits of $\vec{B}$ are in state $|0\rangle$ of their local Z bases, and all qubits of $\vec{C}$ are in state $|1\rangle$ of their respective Z bases. The non-vanishing correlation tensor components in xy plane, which involve only $(M+1)$-partite correlations are $T_{x\vec{w}\vec{0}} = \pm\sin 2\alpha$, $T_{x\vec{0}\vec{w}} = \pm 1$, and $T_{y\vec{0}\vec{v}} = -\cos 2\alpha$, where $\vec{w}$ contains even number of y indices, other indices being x, and $\vec{v}$ contains odd number of y indices, other indices again being x. There are $\sum_{k=1}^{\lfloor M/2 \rfloor}\binom{M}{2k} = 2^{M-1}$ correlation tensor elements of each type and consequently

$$\mathcal{L}_{A\vec{B}}^2 = 2^{M-1}\sin^2 2\alpha, \quad \mathcal{L}_{A\vec{C}}^2 = 2^{M-1}(1+\cos^2 2\alpha). \tag{14}$$

Therefore, the bound of (12) is always achieved and all allowed values of $\mathcal{L}_{A\vec{B}}$ and $\mathcal{L}_{A\vec{C}}$ can be attained either by the state (13) or the state with the role of qubits $\vec{B} \leftrightarrow \vec{C}$ interchanged.

This multipartite scenario has the feature of monogamy in the sense that if one set of observers maximally violates a Bell inequality, the other set observes vanishing Bell parameters. However, the trade-off relation is in stark contrast with the bipartite case. It is impossible for two sets of observers to violate bipartite Bell inequalities, whereas in Eq. (14) both terms can be bigger than one indicating that both sets of observers can simultaneously violate the multipartite inequality.

The underlying reason why the trade-off relation (12) allows for violation by both $A\vec{B}$ and $A\vec{C}$ is the fact that sets of anti-commuting operators of the Bell parameters can contain at most four elements. Now we present a much stronger monogamy related to the graph in Fig. 1d. Consider M-partite Bell inequalities corresponding to different paths from the root of the graph to its leaves ($M = 3$ in Fig. 1d). There are 2^{M-1} such inequalities and we shall prove that their quantum mechanical values obey

$$\mathcal{L}_1^2 + \cdots + \mathcal{L}_{2^{M-1}}^2 \leq 2^{M-1}, \tag{15}$$

where $\mathcal{L}_j$ is the quantum value for the j-th Bell parameter in the graph. To prove this, we construct 2^{M-1} sets of anti-commuting operators, each set containing 2^M elements, such that they exhaust all correlation tensor elements which enter the bound of the left-hand side of (15) after application of condition (5). The construction also uses the graph of the binary tree. We begin at the root, to which we associate a set of two anti-commuting operators, X and Y, for the corresponding qubit. A general rule now is that if we move up in the graph from qubit A to qubit B we generate two new anti-commuting operators by placing X or Y at position B to the operator which had X at position A. Similarly, if we move down in the graph to qubit C we generate two new anti-commuting operators by placing X or Y at position C to the operator which contained Y at position A. For example, starting from the set of operators (X, Y) by moving up we obtain $(XX\mathbb{1}, XY\mathbb{1})$, and by moving down we have $(Y\mathbb{1}X, Y\mathbb{1}Y)$. The next sets of operators are $(XX\mathbb{1}X\mathbb{1}\mathbb{1}\mathbb{1}, XX\mathbb{1}Y\mathbb{1}\mathbb{1}\mathbb{1})$, $(XY\mathbb{1}\mathbb{1}X\mathbb{1}\mathbb{1}, XY\mathbb{1}\mathbb{1}Y\mathbb{1}\mathbb{1})$, $(Y\mathbb{1}X\mathbb{1}\mathbb{1}X\mathbb{1}, Y\mathbb{1}X\mathbb{1}\mathbb{1}Y\mathbb{1})$ and $(Y\mathbb{1}Y\mathbb{1}\mathbb{1}\mathbb{1}X, Y\mathbb{1}Y\mathbb{1}\mathbb{1}\mathbb{1}Y)$ if we move from the root: up up, up down, down up and down down, respectively. By following this procedure in the whole graph we obtain a set of 2^M mutually anti-commuting operators. According to this algorithm the anti-commuting operators can be grouped in pairs having the same Pauli operators except for the qubits of the last step (the leaves of the graph). There are 2^{M-1} such pairs corresponding to distinct combinations of tensor products of X and Y operators on $M-1$ positions. Importantly, in different operators these positions are different and to generate the whole set of operators entering the bound we have to perform suitable permutations of positions. Such permutations always exist and they do not affect anti-commutativity. Finally we end up with the promised 2^{M-1} sets of 2^M anti-commuting operators each, which according to our theorem give the bound of (15).

The inequality (15) is stronger than the trade-off relation (12) in the sense that it does not allow simultaneous violation of all the inequalities of its left-hand side. All other patterns of violations are possible as we now show. Choose any number, m, of Bell inequalities, i.e. paths in the Fig. 1d. Altogether they involve n parties which share the following quantum state

$$|\psi_n\rangle = \frac{1}{\sqrt{2}}|\underbrace{0\ldots0}_{n}\rangle + \frac{1}{\sqrt{2m}}\sum_{j=1}^{m}|0\ldots0\underbrace{1\ldots1}_{\mathcal{P}_j}0\ldots0\rangle, \tag{16}$$

where $\mathcal{P}_j$ denotes parties involved in the j-th Bell inequality. Note that all states under the sum are orthogonal as they involve different parties. The only non-vanishing components of the correlation tensor of this state have even number of y indices for the parties involved in the Bell inequalities. Squares of all these components are equal to $\frac{1}{m}$ which gives $\mathcal{L}_j^2 = \frac{2^{M-1}}{m}$ for each Bell inequality $j = 1, \ldots, m$. Therefore, all m Bell inequalities are violated as soon as $m < 2^{M-1}$. Moreover, the sum of these m Bell parameters saturates the bound of (15) and therefore independently of the state shared by other parties the remaining Bell parameters of (15) all vanish.

In conclusion, we have derived monogamy of multipartite Bell inequality violations which are all quadratic functions of Bell parameters. As such these relations are stronger than those following from no-signaling principle alone, which are linear in Bell parameters [4–7]. Indeed, some of our monogamies are tight in the sense that they precisely identify the set of Bell violations allowed by quantum theory. Our proofs are within quantum formalism and utilize the bounds imposed by the complementarity principle. It would be interesting to see if the Bell violation trade-offs can be derived without using quantum formalism. A plausible candidate for this task is the principle of information causality [19].

Acknowledgements. This research is supported by the National Research Foundation and Ministry of Education in Singapore. WL is supported by the EU program Q-ESSENCE (Contract No.248095), the MNiSW Grant no. N202 208538 and by the Foundation for Polish Science.

* Electronic address: phykd@nus.edu.sg

[1] J. S. Bell, Physics **1**, 195 (1964).

[2] V. Scarani and N. Gisin, Phys. Rev. Lett. **87**, 117901 (2001); Phys. Rev. A **65**, 012311 (2001).

[3] B. Toner and F. Verstraete, quant-ph/0611001.

[4] J. Barrett, N. Linden, S. Massar, S. Pironio, S. Popescu, and D. Roberts, Phys. Rev. A **71**, 022101 (2005)

[5] Ll. Masanes, A. Acin, and N. Gisin, Phys. Rev. A **73**, 012112 (2006).

[6] B. Toner, Proc. R. Soc. A **465**, 59 (2009).

[7] M. Pawłowski and Č. Brukner, Phys. Rev. Lett. **102**, 030403 (2009).

[8] R. Ramanathan, T. Paterek, A. Kay, P. Kurzyński, D. Kaszlikowski, in preparation.

[9] S. Wehner and A. Winter, J. Math. Phys. **49**, 062105 (2008).

[10] H. Weinfurter and M. Żukowski, Phys. Rev. A **64**, 010102(R) (2001).
[11] R. F. Werner and M. W. Wolf, Phys. Rev. A **64**, 032112 (2001).
[12] M. Żukowski and Č. Brukner, Phys. Rev. Lett. **88**, 210401 (2002).
[13] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, Phys. Rev. Lett. **23**, 880 (1969).
[14] V. Coffman, J. Kundu, and W. Wootters, Phys. Rev. A **61**, 052306 (2000).
[15] T. J. Osborne and F. Verstraete, Phys. Rev. Lett. **96**, 220503 (2006).
[16] M. Koashi and A. Winter, Phys. Rev. A **69**, 022309 (2004).
[17] T. Hiroshima, G. Adesso, and F. Illuminati, Phys. Rev. Lett. **98**, 050503 (2007).
[18] N. D. Mermin, Phys. Rev. Lett. **65**, 1838 (1990).
[19] M. Pawłowski, T. Paterek, D. Kaszlikowski, V. Scarani, A. Winter, and M. Żukowski, Nature **461**, 1101 (2009).