Fisher information, spin squeezing, and multiparticle entanglement

Philipp Hyllus[1], Wiesław Laskowski[2,3,4], Roland Krischek[3,4], Christian Schwemmer[3,4], Witlef Wieczorek[3,4,5], Harald Weinfurter[3,4], Luca Pezzé[6], and Augusto Smerzi[1],
[1]*INO-CNR BEC Center and Dipartimento di Fisica, Università di Trento, I-38123 Povo, Italy*
[2]*Institute of Theoretical Physics and Astrophysics, University of Gdańsk, PL-80-952 Gdańsk, Poland*
[3]*Fakultät für Physik, Ludwig-Maximilians Universität München, D-80799 München, Germany*
[4]*Max-Planck Institut für Quantenoptik, D-85748 Garching, Germany*
[5]*Faculty of Physics, University of Vienna, A-1090 Vienna, Austria*
[6]*Laboratoire Charles Fabry, Institut d'Optique, 91127 Palaiseau, France*
(Dated: June 24, 2010)

The Fisher information F gives a limit to the ultimate precision that can be obtained in a phase estimation protocol. It has been shown recently that F cannot exceed the number of particles in a linear two-mode interferometer if the input state is separable. This implies that with such input states the shot-noise limit is the ultimate limit of precision. We extend this result by constructing bounds on F for several multiparticle entanglement classes. We further compute similar bounds on the Fisher information averaged over all possible linear interferometers $\bar{F}$. We show that these criteria detect different sets of states and illustrate their strengths by considering several examples. For instance, the criterion based on $\bar{F}$ is able to detect certain bound entangled states. Finally, we comment on the relation to bounds on the spin squeezing parameter for multipartite entangled states obtained previously, pointing out the connection between the Fisher information, spin squeezing, and multipartite entanglement.

PACS numbers: 03.67.-a, 03.67.Mn, 06.20.Dk, 42.50.St

I. INTRODUCTION

Entanglement is a distinguishing feature of quantum theory that allows to perform several tasks better than it is possible with classical means [1]. Therefore, it is important to study the connection between the entanglement of quantum states and the usefulness for specific applications. One example of such a task is phase estimation as done in quantum interferometry [2], where the connection between entanglement and phase sensitivity has been investigated recently [3–6]. In general, the structure of the set of entangled bipartite quantum states is understood quite well, while the classification and quantification of multipartite quantum states is less developed [7–9]. In particular, if a mixed quantum state ρ is given, it is typically a difficult question to answer how many particles are entangled in the state [10–16]. Commonly applied criteria to distinguish between different entanglement classes include entanglement witnesses [9] and Bell inequalities [17–20]. Recently, other approaches have led to criteria which can be evaluated directly from elements of the density matrix [21, 22].

In this manuscript, we introduce two sets of novel criteria which can distinguish between different entanglement classes, which are deeply connected to phase estimation. This extends the previous work on the connection between entanglement and phase sensitivity mentioned above. The criteria are based on the Fisher information and linear two-mode interferometers. The first set of criteria is based directly on the quantum Fisher information, which is the optimum of the Fisher information over all output measurements of the phase estimation protocol [23]. We compute the optimal values for different entanglement classes. The second set of criteria is based on the quantum Fisher information averaged over the direction of the interferometer to be defined below. We show that the sets of states that the criteria detect are different and not contained in each other. We consider several examples in order to assess the strength of the criteria. Finally, we comment on the relation of the criteria to the spin squeezing parameter [24] and on bounds thereof for multipartite entanglement classes obtained in [25, 26].

The article is organized as follows. We start by introducing the basic concepts related to general phase estimation protocols, linear two-mode interferometers, and the classification of multiparticle entanglement in Section II. Then we derive the entanglement criteria based on the quantum Fisher information and on the average quantum Fisher information in Section III, where we also make a first comparison of the two. In Section IV, we apply the criteria to several families of entangled states. Finally, we investigate the relation to spin squeezing inequalities in Section V. We conclude in Section VI.

II. BASIC CONCEPTS

In a general phase estimation scenario, an input state ρ is transformed into the state $\rho(\theta)$ depending on a phase shift θ. Then, a general positive operator valued measurement (POVM) with elements $\{\hat{E}_\mu\}_\mu$ is performed. This procedure may be repeated m times, and based on the results collected in the vector $\vec{\mu}$, the phase shift is estimated by an estimator $\theta_{\rm est}(\vec{\mu})$. If the estimator is unbiased, *i.e.*, $\langle\theta_{\rm est}\rangle=\theta$, then its minimal standard deviation is limited by the Cramer-Rao bound [27, 28]

$$\Delta\theta_{\rm est}\geq\frac{1}{\sqrt{mF}},\tag{1}$$

where F is the Fisher information, which is defined as

$$F=\sum_{\mu}\frac{1}{P(\mu|\theta)}[\partial_\theta P(\mu|\theta)]^2,\tag{2}$$

where $P(\mu|\theta)={\rm Tr}[\rho(\theta)\hat{E}_\mu]$. A maximum likelihood estimator saturates the Cramer-Rao bound in the central limit, for a sufficiently large m [29]. In this sense, F quantifies the asymptotic usefulness of a quantum state for phase estimation if the phase transformation and the output measurement are fixed. Maximizing F over all possible POVMs leads to the so-called quantum Fisher information F_Q. If the phase shift is generated by an operator $\hat{H}$, and for a mixed input state $\rho=\sum_k\lambda_k|k\rangle\langle k|$, then it is given by [23]

$$F_Q=2\sum_{k,l}\frac{(\lambda_k-\lambda_l)^2}{\lambda_k+\lambda_l}|\langle l|\hat{H}|k\rangle|^2.\tag{3}$$

For pure input states this reduces to $F_Q=4\langle(\Delta\hat{H})^2\rangle$ [23].

In general linear two-mode interferometers and input states of N particles, $\hat{H}$ is given by a collective spin operator $\hat{J}_{\vec{n}}=\vec{n}\cdot\vec{\hat{J}}$, where $\hat{J}_i=\frac{1}{2}\sum_{k=1}^N\sigma_i^{(k)}$ and $\sigma_i^{(k)}$ is the i-th Pauli matrix acting on particle k. The operators $\hat{J}_x$, $\hat{J}_y$, and $\hat{J}_z$ fulfil the commutation relations of spin. We will use the symmetric eigenstates $|j,\mu\rangle$ of $\hat{J}_z$, where $j=\frac{N}{2}$ and $\mu=-\frac{N}{2},-\frac{N}{2}+1,...,\frac{N}{2}$, and the eigenstates of σ_z defined by $\sigma_z|l\rangle=(-1)^l|l\rangle$, where $l=0,1$, where $|0\rangle$ and $|1\rangle$ are the two states that the interferometer works with. As an example for a linear two-mode interferometer we mention that the Mach-Zehnder interferometer has the generator $\hat{H}=\hat{J}_y$ [30].

For generators $\hat{J}_{\vec{n}}$, there is a direct connection between entanglement and sub shot-noise interferometry. An entangled state of N particles cannot be written as $\rho_{\rm sep}=\sum_j p_j\bigotimes_{k=1}^N|\psi_j^{(k)}\rangle\langle\psi_j^{(k)}|$, where $\{p_j\}$ forms a probability distribution [31]. It has been shown recently that

$$F_Q[\rho_{\rm sep},\hat{J}_{\vec{n}}]\leq N\tag{4}$$

holds [3, 4]. Therefore, the phase uncertainty is bounded by the shot-noise limit

$$\Delta\theta_{\rm est}\geq\frac{1}{\sqrt{mN}}\tag{5}$$

for separable states.

The purpose of this manuscript is to derive similar bounds for multipartite entanglement classes. We consider the following classification of multiparticle entanglement from Ref. [32] (see [11, 33] for alternative classifications): a pure state of N particles is k-partite entangled if there are at most groups of k particles which are fully entangled within the state. We denote such a state as $|\psi_{k-{\rm ent}}\rangle$. This is *producible* by interactions between k particles at most. A mixed state is k-partite entangled if it can be written as a mixture of k-partite entangled pure states, *i.e.*, $\rho_{k-{\rm ent}}=\sum_k p_k|\psi_{k-{\rm ent}}\rangle\langle\psi_{k-{\rm ent}}|$. A separable state is 1-partite entangled in this notation. Note that a decomposition of a $k<N$-partite entangled state of N particles may contain states where different sets of particles are entangled. Let us illustrate this by considering $N=3$. A state $|\psi_{1-{\rm ent}}\rangle=|\phi\rangle_1\otimes|\varphi\rangle_2\otimes|\chi\rangle_3$ is fully separable, a state $|\psi_{2-{\rm ent}}\rangle=|\phi\rangle_{12}\otimes|\chi\rangle_3$ which cannot be written as $|\psi_{1-{\rm ent}}\rangle$ is 2-partite entangled, and a state $|\psi_{3-{\rm ent}}\rangle$ which cannot be written as a 2-partite entangled state is 3-partite entangled.

III. CRITERIA FOR MULTIPARTITE ENTANGLEMENT FROM THE QUANTUM FISHER INFORMATION

Now we are in a position to derive the desired bounds. We start by deriving bounds on the quantum Fisher information $F_Q[\rho_{k-{\rm ent}};\hat{J}_{\vec{n}}]$ for k-partite entangled states and an arbitrary direction $\vec{n}$. Then, we derive bounds on the quantum Fisher information for a generator $\hat{J}_{\vec{n}}$, averaged over all directions $\vec{n}$. At the end of this section, we show for $k=1$ and $k=N-1$ the sets of states that the two criteria detect are different and not contained in each other.

A. Criteria from $\mathbf{F_Q}$

We directly present the result, the derivation follows afterwards.

Observation 1 ($\mathbf{F_Q^{k+1}}$ criterion). *For k-partite entangled states and a linear two-mode interferometer with a generator $\hat{J}_{\vec{n}}$ of an arbitrary direction $\vec{n}$, the quantum Fisher information is bounded by*

$$F_Q[\rho_{k-{\rm ent}};\hat{J}_{\vec{n}}]\leq sk^2+r^2,\tag{6}$$

where $s=\lfloor\frac{N}{k}\rfloor$ is the largest integer smaller than or equal to $\frac{N}{k}$ and $r=N-sk$. Hence a violation of the bound (6) proves $(k+1)$-partite entanglement.

Proof. The basic ingredients of the derivations are the following: (i) The sets of states introduced above are convex. (ii) The Fisher information is convex in the states, *i.e.*, for the generators we consider, $F[p\rho_1+(1-p)\rho_2;\hat{J}_{\vec{n}}]\leq pF[\rho_1;\hat{J}_{\vec{n}}]+(1-p)F[\rho_2;\hat{J}_{\vec{n}}]$ for $p\in[0,1]$ [34]. Since the quantum Fisher information is equal to the Fisher information for a particular measurement, this holds also for F_Q. It follows that the maximum of F_Q for a fixed $\vec{n}$ and k-partite entangled mixed states is reached on the pure k-partite entangled states. In this case, $F_Q[|\psi\rangle;\hat{J}_{\vec{n}}]=4\langle(\Delta\hat{J}_{\vec{n}})^2\rangle_{|\psi\rangle}$ as mentioned above. (iii) It is easy to see that for a product state $|\phi_A\rangle\otimes|\chi\rangle_B$, $4\langle(\Delta[\hat{J}_{\vec{n}}]_{AB})^2\rangle_{|\phi\rangle_A\otimes|\chi\rangle_B}=$

$4\langle(\Delta[\hat{J}_{\vec{n}}]_A)^2\rangle_{|\phi\rangle_A} + 4\langle(\Delta[\hat{J}_{\vec{n}}]_B)^2\rangle_{|\chi\rangle_B}$. Here $[\hat{J}_{\vec{n}}]_{AB}$ acts on all the particles while $[\hat{J}_{\vec{n}}]_A$ acts on the particles of $|\psi\rangle_A$ only and in analogy for $[\hat{J}_{\vec{n}}]_B$. (iv) For a state with N particles, $4\langle(\Delta\hat{J}_{\vec{n}})^2\rangle \leq N^2$ holds [3]. The inequality is saturated uniquely by the state $|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}(|0\rangle^{\otimes N} + |1\rangle^{\otimes N})$, known as the GHZ [35] or NOON [36] state.

Let us now consider the state $|\psi\rangle = \bigotimes_{j=1}^{M}|\psi_j\rangle$, where $|\psi_j\rangle$ is a state of N_j particles, and $\sum_j N_j = N$. If we allow for at most k particles to be entangled, then $N_j \leq k$. Since $(N_1+1)^2+(N_2-1)^2 \geq N_1^2+N_2^2$ if $N_1 \geq N_2$, the quantum Fisher information is increased by making the N_j as large as possible. Hence the maximum is reached by choosing $N_j = k$ for $s = \lfloor\frac{N}{k}\rfloor$ states, where $\lfloor x\rfloor$ is the largest integer smaller or equal to x, and by collecting the remaining $r = N - sk$ particles in the last state. Therefore, for k-partite entangled states, the quantum Fisher information is bounded by Eq. (6). □

The bound can be saturated by a product of s GHZ states of k particles and another GHZ state of r particles. We point out that for $k = 1$ we recover the bound (4) for separable states, while for $k = N-1$, the bound is

$$F_Q[\rho_{(N-1)-\mathrm{ent}}; \hat{J}_{\vec{n}}] \leq (N-1)^2 + 1. \tag{7}$$

These results could have been obtained directly by using the Wigner-Yanase information I [37]. The bound (6) has been derived previously for $4I$ in Ref. [10], and directly applies to the quantum Fisher information since I is convex in the states and agrees with the Fisher information on pure states, $F[|\psi\rangle; \hat{J}_{\vec{n}}] = 4I(|\psi\rangle, \hat{J}_{\vec{n}})$. We have presented the derivation here in order to emphasize that the classification relies on GHZ states, and in order to introduce concepts that we will also need in the following.

For a fixed direction $\vec{n}$, the criterion (6) has a clear operational meaning, because if the bound is surpassed, then the state contains $(k+1)$-partite entanglement, and it will enable a better precision when used in the interferometer defined by $\hat{J}_{\vec{n}}$ than all k-partite entangled states. If used as a criterion for multiparticle entanglement, it is therefore advantageous to optimize the direction $\vec{n}$, which can be done analytically as follows: the quantum Fisher information can be written as [6]

$$F[\rho; \hat{J}_{\vec{n}}] = 4\vec{n}^T\Gamma_C\vec{n}. \tag{8}$$

The matrix Γ_C is real and symmetric and has the entries

$$[\Gamma_C]_{ij} = \frac{1}{2}\sum_{l,m}\frac{(\lambda_l-\lambda_m)^2}{\lambda_l+\lambda_m}\langle l|\hat{J}_i|m\rangle\langle m|\hat{J}_j|l\rangle, \tag{9}$$

where the states $|k\rangle$ and the variables λ_k are defined by the eigenvalue decomposition of the input state, $\rho = \sum_k \lambda_k|k\rangle\langle k|$. It follows that $F_Q^{\max}[\rho] \equiv \max_{\vec{n}} F_Q[\rho; \hat{J}_{\vec{n}}] = 4\lambda_{\max}(\Gamma_C)$, where $\lambda_{\max}(\Gamma_C)$ is the maximal eigenvalue of Γ_C [6]. We will always use $F_Q^{\max}[\rho]$ in the following and refer to the bound (6) as F_Q^{k+1} criterion since it allows for the detection of $(k+1)$-entanglement *via* its violation.

B. Criteria from $\bar{\mathbf{F}}_{\mathbf{Q}}$

We will now derive similar bounds for the average quantum Fisher information, which we define as

$$\bar{F}_Q[\rho] = \frac{1}{4\pi}\int_{|\vec{n}|^2=1} \mathrm{d}^3\vec{n}\; F_Q[\rho; \hat{J}_{\vec{n}}]. \tag{10}$$

The prefactor ensures normalization in the sense that if $F[\rho; \hat{J}_{\vec{n}}] = F[\rho; \hat{J}_x]$ is independent of the direction $\vec{n}$, then $\bar{F}_Q = F[\rho; \hat{J}_x]$. The quantity $\bar{F}_Q$ has the following meaning: suppose that an experiment implements an interferometer with a fixed phase shift θ, but with no control over the axis $\vec{n}$ of rotation, such that in each run of the experiment, the interferometer operation is given by $\exp[-i\hat{J}_{\vec{n}}\theta]$, with a random direction $\vec{n}$. If all directions appear with equal probability, and for $m \gg 1$ independent repetitions of the experiment, the corresponding quantum Fisher information would be given $m\,\bar{F}_Q$.

The average can be written as $\bar{F}_Q = \frac{1}{\pi}\sum_{i,j}[\Gamma_C]_{i,j}\int_{|\vec{n}|^2=1}\mathrm{d}^3\vec{n}\;\; n_i n_j$. Evaluating the integrals leads to

$$\bar{F}_Q = \frac{4}{3}\mathrm{Tr}[\Gamma_C] = \frac{1}{3}\Big(F_Q[\rho; \hat{J}_x] + F_Q[\rho; \hat{J}_y] + F_Q[\rho; \hat{J}_z]\Big). \tag{11}$$

We would like to determine bounds on $\bar{F}_Q$ for k-partite entangled states in analogy to the bounds that we found for F_Q. Again, we directly state the results and derive them afterwards.

Observation 2 ($\bar{\mathbf{F}}_{\mathbf{Q}}^{\mathbf{k+1}}$ criteria). *For k-partite entangled states and a linear two-mode interferometer with a generator $\hat{J}_{\vec{n}}$ with an arbitrary direction $\vec{n}$, the average quantum Fisher information is bounded by*

$$\bar{F}_Q[\rho_{k-\mathrm{ent}}] \leq \frac{1}{3}[s(k^2+2k-\delta_{k,1}) + r^2 + 2r - \delta_{r,1}], \tag{12}$$

where $s = \lfloor\frac{N}{k}\rfloor$ and $r = N - sk$. Hence a violation of the bound (12) proves $(k+1)$-partite entanglement. For separable states, corresponding to $k = 1$, the bound becomes

$$\bar{F}_Q[\rho_{\mathrm{sep}}] \leq \frac{2}{3}N. \tag{13}$$

The maximal value for any quantum state is given by

$$\bar{F}_Q \leq \frac{1}{3}[N^2 + 2N]. \tag{14}$$

Proof. Let us first prove Eq. (14). Since $\bar{F}_Q$ can be written as the sum of three quantum Fisher informations, it is also convex in the states. Therefore, the maximum is again reached for pure states. Hence

$\bar{F} \leq \frac{4}{3}\max_{|\psi\rangle}[\langle\vec{\hat{J}}^2\rangle_{|\psi\rangle} - \langle\vec{\hat{J}}\rangle^2_{|\psi\rangle}] \leq \frac{4}{3}j(j+1)$. This leads to Eq. (14) since $j = \frac{N}{2}$. The last inequality follows because $\langle\vec{\hat{J}}\rangle^2_{|\psi\rangle} \geq 0$ and

$$\langle\vec{\hat{J}}^2\rangle_{|\psi\rangle} \leq j(j+1) \tag{15}$$

holds in general, while equality is reached by the symmetric states of N particles.

For the state $|\psi\rangle = \bigotimes_{j=1}^{M}|\psi_j\rangle$ introduced in the proof of Observation 1, the average quantum Fisher information is given by $\bar{F}_Q = \frac{4}{3}\sum_{j=1}^{M}[\langle\vec{\hat{J}}_j^2\rangle_{|\psi_j\rangle} - \langle\vec{\hat{J}}_j\rangle^2_{|\psi_j\rangle}] \leq \frac{1}{3}\sum_{j=1}^{M}[N_j^2 + 2N_j - 4\langle\vec{\hat{J}}_j\rangle^2_{|\psi_j\rangle}]$, where $\vec{\hat{J}}_j$ is the vector of collective spin operators acting on the particles contained in state $|\psi_j\rangle$. The inequality is due to Eq. (15). In the same way as it was for F_Q, it is advantageous to increase the N_j as much as possible. This is true even though if $N_j = 1$ then $\bar{F}_Q$ is reduced by $\frac{1}{3}$ since $\langle\vec{\hat{J}}\rangle^2_{|\psi_j\rangle} = \frac{1}{4}$ in this case. For $k \in [1, N]$, we obtain the bound (12), where $s = \lfloor\frac{N}{k}\rfloor$ and $r = N - sk$ as above, and we obtain Eq. (13) for $k = 1$. □

If a state violates the bound (12), then it is $(k+1)$-entangled. Therefore, we refer to this bound as $\bar{F}_Q^{k+1}$ criterion. Let us note that the bound for $k = N-1$ is

$$\bar{F}_Q[\rho_{(N-1)-\mathrm{ent}}] \leq \frac{1}{3}[N^2+1]. \tag{16}$$

C. $\mathbf{F_Q}$ criteria vs. $\mathbf{\bar{F}_Q}$ criteria

Since both the F_Q^{k+1} criteria and the $\bar{F}_Q^{k+1}$ criteria seem to be very much related, it is natural to ask whether or not one set of criteria is stronger than the other. We examine this question for $k=1$ and $k = N-1$ only, since the other possible values of k depend on N. The result is as follows.

Observation 3. *For $k = 1$ and $k = N-1$, the F_Q^{k+1} criteria and the $\bar{F}_Q^{k+1}$ criteria detect different sets of states which are not contained in each other.*

In order to prove this, we consider states of the form

$$\rho(p) = p|\psi\rangle\langle\psi| + (1-p)\frac{\mathbb{1}}{2^N}, \tag{17}$$

mixtures of a pure state and the totally mixed state. It can be shown directly from Eq. (9) that

$$\Gamma_C[\rho(p)] = \gamma_{p,N}\Gamma_C[|\psi\rangle], \quad \gamma_{p,N} = \frac{p^2 2^{N-1}}{p(2^{N-1}-1)+1} \tag{18}$$

holds. Let us first consider the case $k = N-1$. The criteria (7) and (16) can be rewritten as $\gamma_{p,N} \leq \alpha_N^{(1,2)}$, where $\alpha_N^{(1)} = [(N-1)^2+1]/F_Q^{\max}[|\psi\rangle]$ and $\alpha_N^{(2)} = [N^2+1]/4\mathrm{Tr}(\Gamma_C[|\psi\rangle])$, respectively. In order to violate the criteria,

$$p > \alpha_N^{(1,2)}\frac{1-2^{1-N}}{2}\left[1+\sqrt{1+\frac{1}{\alpha_N^{(1,2)}}\frac{2^{3-N}}{(1-2^{1-N})^2}}\right]. \tag{19}$$

This function is strictly monotonically increasing with $\alpha_N^{(1,2)}$. If, for instance, $\alpha_N^{(1)} < \alpha_N^{(2)}$, then the F_Q^N criterion detects the states as multipartite entangled already for a smaller value of p than the $\bar{F}_Q^N$ criterion. Therefore, we can prove the claim by comparing the α coefficients for different states $|\psi\rangle$.

We employ the GHZ state introduced above and the twin-Fock state $|\mathrm{TF}_N\rangle = |\frac{N}{2}, 0\rangle$ with an equal number of particles in both states [38], which is well known to provide sub shot-noise phase sensitivity [38–40]. For the GHZ states, the eigenvalues of $4\Gamma_C$ are N^2, N, and N, while for the twin-Fock states, the eigenvalue are $\frac{N^2}{2}+N$ appears twice, and the third eigenvalue vanishes [6]. While the GHZ state achieves the highest value of the Fisher information possible for $\vec{n} = \hat{z}$, where $\hat{z}$ is the unit vector pointing in the z-direction, it is shot-noise limited for any direction in the $x-y$ plane [6]. In contrast, the twin-Fock state surpasses the shot-noise limit for any direction in the $x-y$-plane, and does not provide any information on the phase when the z-direction is used [6, 41, 42]. We obtain $F_Q^{\max} = N^2$ and $\bar{F}_Q = N^2 + 2N$ for the GHZ state and $F_Q^{\max} = \frac{N^2}{2} + N$, and $\bar{F}_Q = N^2 + 2N$ for the twin-Fock state. Inserting these values into $\alpha_N^{(1,2)}$, we directly see that $\alpha_N^{(1)}(\mathrm{GHZ}) < \alpha_N^{(2)}(\mathrm{GHZ})$ while $\alpha_N^{(1)}(\mathrm{TF}) > \alpha_N^{(2)}(\mathrm{TF})$. This proves the claim for $k = N-1$.

We can apply the same reasoning to the case $k = 1$, where we have to adapt $\alpha_N^{(1)} \to \beta^{(1)} = N/F_Q^{\max}[|\psi\rangle]$ and $\alpha_N^{(2)} \to \beta^{(2)} = N/2\mathrm{Tr}(\Gamma_C[|\psi\rangle])$. For the GHZ state, we obtain as before that $\beta^{(1)}(\mathrm{GHZ}) < \beta^{(2)}(\mathrm{GHZ})$ while for the twin-Fock state $\beta^{(1)}(\mathrm{TF}) = \beta^{(2)}(\mathrm{TF})$ holds. However, we can construct an example where the $\bar{F}_Q^2$ criterion is stronger. We observe that $\beta^{(1)} > \beta^{(2)} \Leftrightarrow 2\mathrm{Tr}(\Gamma_C[|\psi\rangle]) > F_Q^{\max}[|\psi\rangle]$. An interesting class of states which fulfills this condition is defined by $4\Gamma_C = c_N\mathbb{1}$. These states have the same value $F_Q = c_N = \bar{F}_Q$ for *any* direction $\vec{n}$. One way of constructing such states is by considering a symmetric state $|\psi\rangle = \sum_k \alpha_k|j, m_k\rangle$, and by choosing the α_k and m_k such that $\langle\vec{\hat{J}}\rangle = 0$ and $\langle\hat{J}_x^2\rangle = \langle\hat{J}_y^2\rangle = \langle\hat{J}_z^2\rangle$. The condition $|m_k - m'_k| > 2$ for all k and k' can be used to ensure $\langle\hat{J}_x\rangle = \langle\hat{J}_y\rangle = 0$. We just mention the example $\sqrt{\frac{1}{3}}|2,2\rangle + \sqrt{\frac{2}{3}}|2,-1\rangle$ for $N = 4$, with $\Gamma_C = 2\mathbb{1}$, and hence $F_Q^{\max} = \bar{F}_Q = 8$. This concludes the proof of our claim.

The last example is interesting because it shows that there are states that cannot beat the shot-noise limit for a fixed direction $\vec{n}$, but that perform better than a separable state on average. We consider various interesting states in the following section.

Criterion	detected 2-ent. [%]	detected 3-ent. [%]
F_Q^2	94.32	-
$\bar{F}_Q^2$	98.38	-
$\mathcal{W}$	-	18.99
DME	-	80.63
DME'	-	82.61
F_Q^3	-	22.93
$\bar{F}_Q^3$	-	27.99

TABLE I: Percentage of detected 2-partite and 3-partite entangled pure three-qubit states. See text for details. DME' the whole family of DME conditions, which is obtained by permuting the qubits of the state.

Criterion	detected DME [%]	detected DME' [%]
$\mathcal{W}$	50.56	12.27
F_Q^3	19.45	4.77
$\bar{F}_Q^3$	13.14	3.25

TABLE II: Percentage of 3-partite entangled states which are detected by the entanglement witness, the criterion F_Q^3 and the crition $\bar{F}_Q^3$. In the middle column, only states violating the DME condition (20) have been generated, while in the last column, also states violating any of the other DME conditions obtained by permutations of the particles have been generated.

IV. EXAMPLES

A. Pure states of 3 particles

In order to get an impression of the strength of the criteria, we randomly choose a three-qubit state $|\psi\rangle$ and analyze it using various criteria. First, we evaluate the criteria F_Q^2 and $\bar{F}_Q^2$ which detect entanglement. Further, we compare several criteria detecting multiparticle entanglement: (i) the entanglement witness $\mathcal{W} = \frac{1}{2}\mathbb{1} - |\mathrm{GHZ}\rangle\langle\mathrm{GHZ}|$, which has a positive expectation value for all 2-partite entangled states [43], (ii) the density matrix element condition (DME) which states that

$$|\rho_{18}| \leq \sqrt{\rho_{22}\rho_{77}} + \sqrt{\rho_{33}\rho_{66}} + \sqrt{\rho_{44}\rho_{55}} \tag{20}$$

for all 2-entangled states (ρ_{ij} denote coefficients of a given density matrix $\rho = |\psi\rangle\langle\psi|$) [44], and (iii) the multipartite criteria F_Q^3 and $\bar{F}_Q^3$.

To generate a random pure state [45], we take a vector of a random unitary matrix distributed according to the Haar measure on U(8):

$$\begin{aligned} |\psi\rangle \; = \; & (\cos\alpha_7, \cos\alpha_6 \sin\alpha_7 e^{i\phi_7}, \cos\alpha_5 \sin\alpha_6 \sin\alpha_7 e^{i\phi_6}, \\ & \ldots, \sin\alpha_1 \cdots \sin\alpha_7 e^{i\phi_1}), \end{aligned} \tag{21}$$

where $\alpha_i \in [0, \pi/2]$ and $\phi_k = [0, 2\pi)$. The parameters are drawn with the probability densities: $P(\alpha_i) = i \sin(2\alpha_i)(\sin\alpha_i)^{2i-2}$ and $P(\phi_i) = 1/2\pi$. The calculations were performed for a set of 10^6 states. The results are presented in Tab. I. The averaged criteria seem to detect more states in general. It is surprising that the witness condition detects nearly as many states as the criteria F_Q^3 and $\bar{F}_Q^3$.

B. GHZ-diagonal states

The DME criterion (20) and the criteria obtained thereof by permutations of the qubits completely characterize the GHZ-diagonal states of three qubits [21], which can be written as

$$\frac{1}{\mathcal{N}} \begin{pmatrix} \lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_1 \\ 0 & \lambda_2 & 0 & 0 & 0 & 0 & \mu_2 & 0 \\ 0 & 0 & \lambda_3 & 0 & 0 & \mu_3 & 0 & 0 \\ 0 & 0 & 0 & \lambda_4 & \mu_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu_4 & \lambda_5 & 0 & 0 & 0 \\ 0 & 0 & \mu_3 & 0 & 0 & \lambda_6 & 0 & 0 \\ 0 & \mu_2 & 0 & 0 & 0 & 0 & \lambda_7 & 0 \\ \mu_1 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_8 \end{pmatrix} \tag{22}$$

with real coefficients λ_i and μ_i, where $\mathcal{N}$ is a normalization factor. If $\lambda_i = \lambda_{9-i}$ for $i = 5, 6, 7, 8$, then these states are diagonal in the GHZ-basis $|\psi_{kl}^{\pm}\rangle = \frac{1}{\sqrt{2}}(|0kl\rangle \pm |1\bar{k}\bar{l}\rangle)$, where k and l are equal to 0 or 1, and $\bar{1} = 0$ and $\bar{0} = 1$. We generated 10^6 random states of this form violating Eq. (20) directly, which states $|\mu_1| \leq \lambda_2 + \lambda_3 + \lambda_4$ in this case. The results are shown in Tab. II in the middle column. Then, we generated again 10^6 states violating Eq. (20) or its other forms obtained by permuting the qubits. The results are shown in the right column of Tab. II. The witness criterion detects significantly more states than the criteria based on the Fisher information. Contrary to the case of pure states, the F_Q^3 criterion detects more states than $\bar{F}_Q^3$ in this case. Note that the percentage of detected states reduces significantly for all criteria in the DME' case. The reason is that all criteria work best for the *symmetric* GHZ state, which has the highest weight in the state if only condition (20) is used [21].

The family of states (22) also comprises bound entangled states if $\lambda_1 = \lambda_8 = \mu_1 = 1$ and $\lambda_7 = 1/\lambda_2$, $\lambda_6 = 1/\lambda_3$, $\lambda_5 = 1/\lambda_4$, and $\mu_2 = \mu_3 = \mu_4 = 0$, as long as $\lambda_2\lambda_3 \neq \lambda_4$. Then the states have a positive partial transpose (PPT) [46] for any bipartition of the three particles while still being entangled [33]. It follows that the state cannot be distilled to a GHZ state [11, 47]. We generated again 10^6 random states of these class and applied F_Q^2 and $\bar{F}_Q^2$, but neither criterion detected any of these states. However, we will see presently that $\bar{F}_Q$ is in fact able to detect bound entanglement.

C. Detecting bound entangled states

We consider two families of states where the state has a PPT with respect to some bipartitions, but not with respect to others. Due to the PPT bipartitions it is not possible to distill these states to a GHZ state nontheless [11].

1. Dür state

Interestingly, the $\bar{F}_Q^2$ criterion (13) can reveal entanglement of a bound entangled state introduced by Dür [48]:

$$\rho_N = \frac{1}{N+1}\left(|\phi\rangle\langle\phi| + \frac{1}{2}\sum_{k=1}^{N}(P_k + \bar{P}_k)\right), \tag{23}$$

with $|\phi\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle_1...|0\rangle_N + e^{i\varphi_N}|1\rangle_1...|1\rangle_N\right]$ (φ_N is an arbitrary phase), and P_k being a projector on the state $|0\rangle_1...|1\rangle_k...|0\rangle_N$ with "1" on the kth position ($\bar{P}_k$ is obtained from P_k after replacing "0" by "1" and vice versa). As an example, let us consider cases for $N = 4, 6$ and 8. The corresponding values of $\overline{F}_Q$ are equal to $136/45 \approx 3.02, 104/21 \approx 4.95$ and $560/81 \approx 6.91$, whereas the bounds for entanglement are equal to $8/3 \approx 2.67$, 4 and $16/3 \approx 5.33$, respectively. In the three cases we prove entanglement of the state ρ_N. However, note that in all cases the F_Q^2 criterion does not detect the entanglement. Hence the state is not useful for sub shot-noise interferometry for any direction $\vec{n}$, even though it is more useful than separable states on average over all directions.

2. Generalized Smolin state

As a second example, consider the generalized Smolin state [49]:

$$\rho_N = \frac{1}{2^N}\left(\mathbb{1}^{\otimes N} + (-1)^{N/2}\sum_{i=1}^{3}\sigma_i^{\otimes N}\right). \tag{24}$$

Similarly as above, take the cases of four, six and eight qubits. For these states, the average quantum Fisher information $\overline{F}_Q$ is equal to 4,6 and 8, respectively. The corresponding bounds for entanglement are $8/3 \approx 2.67$, 4 and $16/3 \approx 5.33$ are violated. Again, F_Q^2 does not detect the entanglement of the states.

V. RELATION TO SPIN SQUEEZING INEQUALITIES

A. Spin squeezing parameter and multiparticle entanglement

Let us now compare the criteria based on F_Q and $\bar{F}_Q$ to spin squeezing criteria. Introducing two orthogonal unit vectors $\vec{n}_1$ and $\vec{n}_2$, then the so-called spin-squeezing parameter ξ can be defined as [24]

$$\xi^2 = \frac{N|\langle\hat{J}\rangle_{\vec{n}_1}|^2}{\langle(\Delta\hat{J}_{\vec{n}_2})^2\rangle}. \tag{25}$$

If $\xi < 1$, the corresponding state is useful for sub shot-noise interferometry [24] and entangled, since $\xi^2 \geq 1$ holds for all separable states [25]. This also follows directly from the following inequality

$$F_Q[\rho; \hat{J}_{\vec{n}_3}] \geq \frac{N}{\xi^2}, \tag{26}$$

where $\vec{n}_3$ is a unit vector orthogonal to $\vec{n}_1$ and $\vec{n}_2$ (the vectors may form a right- or left-handed coordinate system) [50].

¿From Eq. (26) it also follows that the F_Q^2 criterion detects all states that the spin-squeezing criterion detects. In fact, it detects more states. For instance, all states with $\langle\vec{\hat{J}}\rangle = 0$, such as the twin Fock state, are not spin squeezed. Nevertheless, the twin Fock state is entangled and useful for sub shot-noise interferometry.

Similarly, bounds on multiparticle entanglement for ξ have been derived as follows [26]. First, the authors regard a single spin-j particle, and compute

$$f_j(\langle\hat{J}_z\rangle) = \min_\rho \langle(\Delta\hat{J}_x)^2\rangle\Big|_{\langle\hat{J}_z\rangle_\rho = \langle\hat{J}_z\rangle}, \tag{27}$$

where the minimization is performed over all states ρ of the spin-j particle which fulfil $\langle\hat{J}_z\rangle_\rho = \langle\hat{J}_z\rangle$. It is then shown that for separable states of s spin-j particles, $\rho_{\text{sep}} = \sum_k p_k \otimes_{l=1}^s |\psi_k^{(l)}\rangle\langle\psi_k^{(l)}|$, where $|\psi_k^{(l)}\rangle$ is a state of a spin-j particle, the inequality

$$\langle(\Delta\hat{J}_x)^2\rangle \geq s f_j\left(\frac{\langle\hat{J}_z\rangle}{s}\right) \tag{28}$$

holds. If the spin-j particles are composed of k spin-$\frac{1}{2}$ particles, then this condition is fulfilled by all k-particle entangled states, and a violation proves the presence of $(k+1)$-particle entanglement. Note that this criterion assumes that in the decomposition of the separable states each state is separable with respect to *the same* partition and that $N = sk$ holds. Both assumptions can be relaxed [51].

Due to Eq. (26), we obtain that for fixed $\langle\hat{J}_z\rangle$, and for $\vec{n}_1 = \hat{z}$, $\vec{n}_2 = \hat{x}$, and $\vec{n}_3 = \hat{y}$, the bounds

$$\begin{aligned}\frac{N}{\xi^2} \;&\leq\; \frac{|\langle\hat{J}_z\rangle|^2}{\langle(\Delta\hat{J}_x)^2_{\rho^*_{k-\text{ent}}}\rangle} \leq F_Q[\rho^*_{k-\text{ent}}; \hat{J}_y] \\ &\leq\; \max_\rho F_Q[\rho; \hat{J}_y]_{\langle\hat{J}_z\rangle_\rho = \langle\hat{J}_z\rangle} \leq nk^2 + r^2,\end{aligned} \tag{29}$$

where $\rho^*_{k-\text{ent}}$ is a k-partite entangled state that minimizes $\langle(\Delta\hat{J}_x)^2\rangle$ for a given $\langle\hat{J}_z\rangle$. The third inequality might be strict due to the limitations of the spin squeezing inequalities, which do not recognize the usefulness of states with

$\langle\vec{\hat{J}}\rangle = 0$, as mentioned above. The last inequality is due to Eq. (6), which bounds the maximal quantum Fisher for *any* k-partite entangled state, not restricted to the subspace with fixed $\langle\hat{J}_z\rangle$.

Note that the condition (28) may recognize a state as $(k+1)$-partite entangled that is not detected by the F_Q^{k+1} criterion (6). This kind of $(k+1)$-entanglement is then not more useful for interferometry than k-partite entanglement. This happens precisely when some of the inequalities are strict.

B. Relation of $\bar{\mathbf{F}}_{\mathbf{Q}}$ and a generalized spin squeezing criterion

Finally, let us point out a connection to a different criterion which detects entanglement, but is not directly related to interferometry. The so-called *generalized* spin squeezing criteria are entanglement criteria based only on first and second moments of collective operators $\hat{J}_{\vec{n}}$ [52, 53]. In particular, for all general mixed separable states the following inequality holds [54]

$$4[(\Delta\hat{J}_x)^2 + (\Delta\hat{J}_y)^2 + (\Delta\hat{J}_z)^2] \geq 2N. \qquad (30)$$

For pure states, the left hand side equals $3\bar{F}$, and therefore, this bound is complementary to the bound (13). However, for pure states, equality holds in both Eq. (30) and Eq. (13), and no contradiction arises.

VI. CONCLUSIONS AND OUTLOOK

We have introduced criteria based on the quantum Fisher information for the detection of entangled states of different multipartite entanglement classes. Additionally, we showed that the quantum Fisher information averaged over all directions on the Bloch sphere is a suitable alternative to detect multipartite entanglement. We considered several examples, showing in particular that the average quantum Fisher information can be used to detect bound entangled states. It remains an interesting open question whether or not there exist bound entangled states which are detected by the quantum Fisher information, since this would imply that such states could be used for sub shot-noise interferometry. We also pointed out the relation of the bounds we derived for the quantum Fisher information to bounds on the spin squeezing parameter for multipartite entanglement classes derived by Sørensen and Mølmer [26].

Acknowledgements. We thank G. Tóth for discussions. We acknowledge support of the EU program Q-ESSENCE (Contract No.248095), the DFG-Cluster of Excellence MAP, and of the EU project QAP. W.L. is supported by the MNiSW Grant no. N202 208538 and by the Foundation for Polish Science (KOLUMB program). The collaboration is a part of a DAAD/MNiSWprogram. W.W. and C.S. acknowledge support by QCCC of the Elite Network of Bavaria.

Note added: Independently from our work, a paper on the relationship between multipartite entanglement and Fisher information is under preparation [55].

[1] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, 2000).
[2] V. Giovannetti, S. Lloyd, and L. Maccone, Science **306**, 1330 (2004).
[3] V. Giovannetti, S. Lloyd, and L. Maccone, Phys. Rev. Lett. **96**, 010401 (2006).
[4] L. Pezzé and A. Smerzi, Phys. Rev. Lett. **102**, 100401 (2009).
[5] The connection between shot-noise limit and separability for a non-fixed number of particles has been discussed in P. Hyllus, L. Pezzé, and A. Smerzi, arXiv:1003.0649.
[6] P. Hyllus, O. Gühne, and A. Smerzi, arXiv:0912.4349.
[7] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Rev. Mod. Phys. **81**, 865 (2009).
[8] M.B. Plenio and S. Virmani, Quant. Inf. Comp. **7**, 1 (2007).
[9] O. Gühne and G. Tóth, Physics Reports **474**, 1 (2009).
[10] Z. Chen, Phys. Rev. A **71**, 052302 (2005).
[11] W. Dür, J.I. Cirac, and R. Tarrach, Phys. Rev. Lett. **83**, 3562 (1999); W. Dür and J.I. Cirac, Phys. Rev. A **61**, 042314 (2000).
[12] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Lett. A **283**, 1 (2001).
[13] P. Wocjan and M. Horodecki, Open Syst. Inf. Dyn. **12**, 331 (2005).
[14] C.S. Yu and H.S. Song, Phys. Rev. A **72**, 022333 (2005).
[15] A.S.M. Hassan and P.S. Joag, QIC **8**, 773 (2008).
[16] M. Hillery, H.T. Dung, and H. Zheng, Phys. Rev. A **81**, 062322 (2010).
[17] D. Collins, N. Gisin, S. Popescu, D. Roberts, and V. Scarani, Phys. Rev. Lett. **88**, 170405 (2002).
[18] M. Seevinck and J. Uffink, Phys. Rev. A **65**, 012107 (2002).
[19] C. Schmid, N. Kiesel, W. Laskowski, W. Wieczorek, and M. Żukowski, and H. Weinfurter, Phys. Rev. Lett. **100**, 200407 (2008).
[20] J.-D. Bancal, C. Branciard, N. Gisin, and S. Pironio, Phys. Rev. Lett. **103**, 090503 (2009).
[21] O. Gühne and M. Seevinck, New Jour. Phys. **12**, 053002 (2010).
[22] M. Huber, F. Minter, A. Gabriel, and B.C. Hiesmayr,

Phys. Rev. Lett. **104**, 210501 (2010).
[23] S. L. Braunstein and C. M. Caves, Phys. Rev. Lett. **72**, 3439 (1994).
[24] D.J. Wineland, J.J. Bollinger, W.M. Itano, and D.J. Heinzen, Phys. Rev. A **50**, 67 (1994).
[25] A. Sørensen, L.-M. Duan, J. I. Cirac, and P. Zoller, Nature **409**, 63 (2001).
[26] A. Sørensen and K. Mølmer, Phys. Rev. Lett. **86**, 4431 (2001).
[27] C. W. Helstrom, *Quantum Detection and Estimation Theory* (Academic Press, New York, 1976).
[28] A. S. Holevo, *Probabilistic and Statistical Aspects of Quantum Theory* (North-Holland, Amsterdam, 1982).
[29] H. Cramer, Mathematical methods of Statistics (Princeton University Press, Princeton, NJ, 1946), esp. pp. 500-504.
[30] B. Yurke, S. L. McCall, and J. R. Klauder, Phys. Rev. A **33**, 4033 (1986).
[31] R. F. Werner, Phys. Rev. A **40**, 4277 (1989).
[32] O. Gühne, G. Tóth, and H. J. Briegel, New J. Phys. **7**, 229 (2005); O. Gühne and G. Tóth, Phys. Rev. A 73, 052319 (2006).
[33] A. Acín, D. Bruß, M. Lewenstein, and A. Sanpera, Phys. Rev. Lett. **87**, 040401 (2001).
[34] M. L. Cohen, IEEE Transactions on Information Theory **14**, 591 (1968).
[35] D. M. Greenberger, M. Horne, and A. Zeilinger, Am. J. Phys. **58**, 1131 (1990).
[36] H. Lee, P. Kok, and J. P. Dowling, J. Mod. Opt. **49**, 2325 (2002).
[37] E.P. Wigner and M.M. Yanase, Proc. Natl. Acad. Sci. U.S.A. **49**, 910 (1963).
[38] M. J. Holland and K. Burnett, Phys. Rev. Lett. **71**, 1355 (1993).
[39] R. A. Campos, C. C. Gerry, and A. Benmoussa, Phys. Rev. A **68**, 023810 (2003).
[40] L. Pezzé and A. Smerzi, Phys. Rev. A **73**, 011801(R) (2006).
[41] W. Wieczorek, R. Krischek, W. Laskowski and H. Weinfurter, in preparation.
[42] W. Wieczorek, PhD thesis, LMU Munich (2009).
[43] M. Bourennane, M. Eibl, Ch. Kurtsiefer, S. Gaertner, H. Weinfurter, O. Gühne, P. Hyllus, D. Bruss, M. Lewenstein, A. Sanpera, Phys. Rev. Lett. **92**, 087902 (2004).
[44] O. Gühne, M. Seevinck, New J. Phys. **12**, 053002 (2010).
[45] K. Życzkowski, H. Sommers, J. Phys. A: Math. Gen. **34** 7111 (2001).
[46] A. Peres, Phys. Rev. Lett. **77**, 1413 (1996); M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Lett. A **223**, 1 (1996).
[47] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. Lett. **80**, 5239 (1998).
[48] W. Dür, Phys. Rev. Lett. **87**, 230402 (2001).
[49] J. A. Smolin, Phys. Rev. A **63**, 032306 (2001); R. Augusiak, P. Horodecki, Phys. Rev. A **73**, 012318 (2006).
[50] This inequality was derived in Ref. [5]. Note that for pure states it follows simply from the commutation relations of the collective spin operators $[\hat{J}_{\vec{n}_1}, \hat{J}_{\vec{n}_2}] = \pm i\hat{J}_{\vec{n}_3}$, where the sign depends on the coordinate system used. Heisenberg's uncertainty relation $\langle(\Delta\hat{J}_{\vec{n}_2})^2\rangle\langle(\Delta\hat{J}_{\vec{n}_3})^2\rangle \geq \frac{|\langle\hat{J}_{\vec{n}_1}\rangle|^2}{4}$ leads to Eq. (26) since $F_Q[\rho;\hat{J}_{\vec{n}_3}] = 4\langle(\Delta\hat{J}_{\vec{n}_3})^2\rangle$ for pure states.
[51] P. Hyllus, W. Laskowski, R. Krischek, C. Schwemmer, W. Wieczorek, H. Weinfurter, L. Pezzé, and A. Smerzi, in preparation.
[52] G. Tóth, C. Knapp, O. Gühne, and H. J. Briegel, Phys. Rev. Lett. **99**, 250405 (2007).
[53] G. Tóth, C. Knapp, O. Gühne, and H. J. Briegel, Phys. Rev. A **79**, 042334 (2009).
[54] G. Tóth, Phys. Rev. A **69**, 052327 (2004).
[55] G. Tóth, in preparation (2010).