Tensor Rank and Stochastic Entanglement Catalysis for Multipartite Pure States

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The tensor rank (also known as generalized Schmidt rank) of multipartite pure states plays an important role in the study of entanglement classifications and transformations. We employ powerful tools from the theory of homogeneous polynomials to investigate the tensor rank of symmetric states such as the tripartite state $|W_3\rangle = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle)$ and its N-partite generalization $|W_N\rangle$. Previous tensor rank estimates are dramatically improved and we show that (i) three copies of $|W_3\rangle$ have a rank of either 15 or 16, (ii) two copies of $|W_N\rangle$ have a rank of $3N - 2$, and (iii) n copies of $|W_N\rangle$ have a rank of $O(N)$. A remarkable consequence of these results is that certain multipartite transformations, impossible even probabilistically, can become possible when performed in multiple-copy bunches or when assisted by some catalyzing state. This effect is impossible for bipartite pure states.

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Multipartite entanglement has attracted increasing attention due to its intriguing properties and potential applications in both quantum information processing and condensed matter physics [1,2]. A central question in the subject concerns the convertibility between different multipartite entangled states by using local operations and classical communications (LOCC). If such a protocol is only stochastic (i.e., occurs with a nonzero probability) then we say that the two states are convertible via stochastic LOCC (SLOCC); when the transformation is reversible, the two states are called SLOCC equivalent. In bipartite systems, SLOCC convertibility is characterized by the Schmidt rank of the state: bipartite $|\psi\rangle$ is SLOCC convertible to $|\phi\rangle$ if and only if the Schmidt rank of $|\psi\rangle$ is no smaller than that of $|\phi\rangle$.

A generalization of the Schmidt rank in multipartite systems and also relevant to SLOCC transformations is the tensor rank. Formally, for states in N-partite quantum systems, each of which is described by a d-dimensional Hilbert space $\mathcal{H}_i$ $(i = 1, \ldots, N)$, the tensor rank $\mathrm{rk}(\psi)$ of a state $|\psi\rangle \in \bigotimes_{\alpha=1}^N \mathcal{H}_\alpha$, defined as the smallest number of product states $\{\bigotimes_{\alpha=1}^N |\phi_i^\alpha\rangle\}_{i=1\ldots rk(\psi)}$ whose linear span contains $|\psi\rangle$. The tensor rank has been extensively studied in algebraic complexity theory [3,4], and while it is easy to compute for $N = 2$ (Schmidt rank), even for $N = 3$, determining the rank of a state is NP hard [5]. This is one reason why SLOCC convertibility in multipartite systems is so challenging.

Despite this general difficulty, the analysis becomes less formidable when certain classes of states are considered such as symmetric states, i.e., those invariant under any permutation of its parties. Recently, symmetric states have received much attention in the study of entanglement measures [6,7] and bound entanglement [8]. Furthermore, the entanglement transformation properties of symmetric states have been investigated [9] and experimental procedures have been designed which use symmetric states in generating families of multiqubit SLOCC equivalent states [10,11].

There is a natural correspondence between symmetric tensors and symmetric polynomials where the theory of homogeneous polynomials can be used to study the latter. As we will explore in greater detail, every homogenous polynomial possesses a quantity called the polynomial rank which is closely related to the tensor rank. The relationship between the two ranks allows for known results on the polynomial rank to be used directly on tensor rank estimations [12]. This method will prove to be quite powerful.

It is easy to see that the tensor rank is an SLOCC monotone: if $|\psi\rangle$ can be transformed into $|\phi\rangle$ via SLOCC, then $\mathrm{rk}(\psi) \geq \mathrm{rk}(\phi)$. In general the converse is not true [13]; however, for any state SLOCC equivalent to the d-level N-partite Greenberger-Horne-Zeilinger (GHZ) state $|\mathrm{GHZ}_N^d\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^d |i\rangle^{\otimes N}$, tensor rank does decide convertibility [14].

Observation 1.—A GHZ-equivalent state $|\psi_{\mathrm{GHZ}}\rangle$ can be SLOCC transformed into $|\phi\rangle$ iff $\mathrm{rk}(\psi_{\mathrm{GHZ}}) \geq \mathrm{rk}(\phi)$. □

Two related types of phenomena studied in entanglement theory are multicopy and entanglement-assisted entanglement transformations. Given a source state $|\psi\rangle$ and a target state $|\phi\rangle$, if there is an integer k such that the transformation of $|\psi\rangle^{\otimes k}$ to $|\phi\rangle^{\otimes k}$ can be achieved by LOCC, then we say that $|\psi\rangle$ can be transformed to $|\phi\rangle$ by multiple-copy entanglement transformation (MLOCC). Similarly, if there is a state $|c\rangle$ such that the transformation of $|\psi\rangle \otimes |c\rangle$ to $|\phi\rangle \otimes |c\rangle$ is possible by LOCC, then we say that $|\psi\rangle$ can be transformed to $|\phi\rangle$ by entanglement-assisted (or catalytic) transformation (ELOCC). The state $|c\rangle$ is called a catalyst for the transformation. For bipartite pure states, it is known that both MLOCC and ELOCC are strictly more powerful than ordinary LOCC [15,16]. In the stochastic versions of multiple-copy and entanglement-assisted transformations (SMLOCC and SELOCC, respectively) we are only concerned with nonvanishing success probability. For bipartite pure states, a transformation is realizable by SLOCC if and only if it is possible by SMLOCC or SELOCC, because of the multiplicativity of the Schmidt rank: $\mathrm{Sch}(\Psi \otimes \Phi) = \mathrm{Sch}(\Psi)\mathrm{Sch}(\Phi)$. Thus, there is no stochastic entanglement catalysis in bipartite systems.

In this Letter, we advance both topics of multipartite tensor ranks and SMLOCC/SELOCC transformations while demonstrating how results of the first have unexpected consequences for the second. As we show, since tensor rank is not multiplicative, there exist instances when the use of multiple copies or a catalyst can increase the conversion probability of some transformation from zero to positive. In the first part of the Letter, we describe the correspondence between homogeneous polynomials and symmetric states, and use it to bound the tensor rank of various multipartite symmetric states. In the second part, we derive some general properties of SMLOCC and SELOCC transformations and then use results from the first part to demonstrate the feasibility of certain SMLOCC and SELOCC transformations when their corresponding SLOCC conversions are impossible.

Homogeneous polynomials and symmetric states.—A symmetric multipartite state is one that is invariant under any permutation of the parties, $|W_3\rangle$ provides a tripartite example. For such a state $|\psi\rangle$, we can ask not only about its tensor rank, but also about its symmetric tensor rank $\mathrm{srk}(\psi)$: the smallest number of symmetric product states $\{|\phi_i\rangle^{\otimes n}\}_{i=1,\ldots,\mathrm{srk}(\psi)}$ to provide an expansion $|\psi\rangle = \sum_{i=1}^{\mathrm{srk}(\psi)} |\phi_i\rangle^{\otimes n}$. To estimate $\mathrm{srk}(\psi)$ [and thus $\mathrm{rk}(\psi)$] we introduce a correspondence between symmetric states and homogeneous polynomials.

A homogeneous polynomial h of order N in d variables $x_1, \ldots, x_d$ is a linear combination of monomials $x^{\underline{j}} = x_1^{j_1}, \ldots, x_d^{j_d}$ (with a multi-index $\underline{j} = j_1, \ldots, j_d$); i.e., it has the form $h = h(x_1, \ldots, x_d) = \sum_{\underline{j}=j_1,\ldots,j_n} a_{\underline{j}} \prod_{i=1}^{d} x_i^{j_i}$, where the sum extends over all multi-indices with $\sum_{i=1}^{d} j_i = N$. Every homogeneous polynomial has a symmetric decomposition $h = \sum_{i=1}^{\mathrm{pr}(h)} (\beta_{1,i}x_1 + \cdots + \beta_{n,i}x_d)^N$, with the minimum number $\mathrm{pr}(h)$ of power terms. We refer to this number as the *polynomial rank* of h. The computation and estimation of polynomial rank is a much-studied problem in algebraic geometry [12,17].

Now, introducing a computational basis $\{|1\rangle, \ldots, |d\rangle\}$ of the d-dimensional local systems $\mathcal{H}_\alpha$, a monomial $x^{\underline{j}}$ is associated with the Dicke state defined as

$$|D(\underline{j})\rangle := \binom{N}{j_1 \ldots j_d}^{1/2} P_{\mathrm{sym}}(|1\rangle^{\otimes j_1} \otimes \cdots \otimes |d\rangle^{\otimes j_d}),$$

where P_{sym} is the projection onto the bosonic (fully symmetric) subspace, $P_{\mathrm{sym}} = \frac{1}{N!}\sum_{\pi \in S_N} U_\pi$, the sum extending over all permutation operators U_π of the N systems. General homogeneous polynomials (symmetric states) are associated by linear extension of the above since monomials (Dicke states) form a basis for the homogeneous polynomials (symmetric states). That is

Observation 2.—Every symmetric state $|\psi\rangle \in (\mathbb{C}^d)^{\otimes N}$ is uniquely associated with a homogeneous polynomial $h(\psi)$ of order N in d variables, and vice versa each homogeneous polynomial h is associated with a symmetric state $|h\rangle$, such that $h(D(\underline{j})) = x^{\underline{j}}$ and $|x^{\underline{j}}\rangle = |D(\underline{j})\rangle$. Under this identification, symmetric tensor rank and polynomial rank are identical: $\mathrm{pr}(h) = \mathrm{srk}(h)$. □

E.g., two copies of $|W_3\rangle$ read $|W_3\rangle^{\otimes 2} = (|003\rangle + |030\rangle + |300\rangle) + (|012\rangle + |021\rangle + |102\rangle + |120\rangle + |201\rangle + |210\rangle)$, which is a sum of two Dicke states having corresponding homogenous polynomials $x_0x_0x_3$ and $x_0x_1x_2$. These have symmetric expansions $x_0x_0x_3 = \frac{1}{6}((x_0 + x_3)^3 - (x_0 - x_3)^3 - 2x_3^3)$ and $x_0x_1x_2 = \frac{1}{24}((x_0 + x_1 + x_2)^3 - (-x_0 + x_1 + x_2)^3 - (x_0 - x_1 + x_2)^3 - (x_0 + x_1 - x_2)^3)$, thus $\mathrm{rk}(W_3^{\otimes 2}) \le \mathrm{srk}(W_3^{\otimes 2}) \le 7$, which is tight [18].

Using observation 1, we prove the following relations between unrestricted and symmetric tensor ranks.

Theorem 3.—(a) For multiqubit Dicke states $|D(m,n)\rangle := P_{\mathrm{sym}}(|0^{\otimes m}, 1^{\otimes n}\rangle)$ with $m \ge n$, $\mathrm{rk}(D(m,n)) = \mathrm{srk}(D(m,n)) = m + 1$, (b) for any N-partite symmetric state $|\psi\rangle$, $\mathrm{rk}(\psi) \le \mathrm{srk}(\psi) \le 2^{N-1}\mathrm{rk}(\psi)$, (c) $\lim_{n\to\infty} \times \sqrt[n]{\mathrm{srk}(\psi^{\otimes n})} = \lim_{n\to\infty}\sqrt[n]{\mathrm{rk}(\psi^{\otimes n})}$.

Proof.—(a) The second equality follows from [17], Cor. 4.5, and it always holds that $\mathrm{rk}(D(m,n)) \le \mathrm{srk}(D(m,n))$. So to prove the first equality, it suffices to show that the lower bound of $\mathrm{rk}(D(m,n))$ equals $m+1$ too. We use induction on n. For $n = 1$, the claim is true [13], and we assume it holds for $n - 1$. Ignoring normalization, we can rewrite the state as $|D(m,n)\rangle = |D(m,n-2)\rangle|11\rangle + |D(m-1,n-1)\rangle \times (|01\rangle + |10\rangle) + |D(m-2,n)\rangle|00\rangle$. Now we perform the global operation $|1\rangle\langle 11| + \frac{1}{2}|0\rangle(\langle 01| + \langle 10|)$ on the last two systems which cannot increase the rank. The resulting $(m+n-1)$-partite state is just the Dicke state $|D(m, n-1)\rangle$ and so $\mathrm{rk}(D(m,n)) \ge \mathrm{rk}(D(m,n-1)) = m + 1$.

(b) Suppose that $|\psi\rangle$ has an optimal product state expansion $\sum_{i=1}^{\mathrm{rk}(\psi)} |A_i\rangle \otimes \cdots \otimes |N_i\rangle$. As $|\psi\rangle$ is symmetric, we have $|\psi\rangle = \sum_{i=1}^{\mathrm{rk}(\psi)} P_{\mathrm{sym}}(|A_i\rangle \otimes \cdots \otimes |N_i\rangle)$. But this is just a sum of $\mathrm{rk}(\psi)$ Dicke states, each one corresponding to the monomial $x_{A_i}, \ldots, x_{N_i}$. From [17], Prop. 11.6, $\mathrm{pr}(x_{A_i}, \ldots, x_{N_i}) \le 2^{N-1}$ which proves the claim.

Part (c) follows directly from (b). □

Three copies of $|W_3\rangle$.—By observation 2, the homogeneous polynomial $h(W_3^{\otimes 3})$ can be written as $\frac{2}{9}(x_0x_1x_6 + x_0x_2x_5 + x_0x_3x_4 + x_1x_2x_4) + \frac{1}{9}x_0^2x_7$. To compute its polynomial rank, we perform the following linear transformations which do not change the polynomial rank: $y_1 = x_1 + x_2 - x_4$, $y_2 = x_1 - x_2 + x_4$, $y_4 = -x_1 + x_2 + x_4$, $z_3 = 1/2(x_3 + x_5)$, $z_5 = 1/2(x_3 + x_6)$, $z_6 = 1/2(x_5 + x_6)$. By using the fact that the polynomial rank is invariant under scalar multiplication, we can remove constant coefficients and obtain $\mathrm{pr}(h(W_3^{\otimes 3})) \leq \mathrm{pr}(x_0 y_1 z_6 - y_1^3) + \mathrm{pr}(x_0 y_2 z_5 - y_2^3) + \mathrm{pr}(x_0 y_4 z_3 - y_4^3) + \mathrm{pr}((y_1 + y_2 + y_4)^3 + x_0^2 x_7) \leq 16$. Here, the inequalities follow from [17], Table 2. With the lower bound $\mathrm{rk}(W_3^{\otimes 3}) \geq 15$ [18], we have

Theorem 4.—(a) $\mathrm{rk}(W_3^{\otimes 3}) = 15$ or 16, (b) $\lim_{n\to\infty} \times \sqrt[n]{\mathrm{rk}(W_3^{\otimes n})} \leq \sqrt[3]{16} \approx 2.52$. □

This improves the previously best bound of $\mathrm{rk}(W_3^{\otimes 3}) \leq 21$ [18]. In particular, Theorem 4 implies that two tripartite GHZ-type states with tensor rank 4 are sufficient to prepare three $|W_3\rangle$ states under SLOCC.

Upper bound on the tensor rank of $|W_N\rangle^{\otimes n}$.—The N-partite W state is defined as the Dicke state $|W_N\rangle = \frac{1}{\sqrt{N}}(|0\ldots 01\rangle + \ldots + |10\ldots 0\rangle) \in (\mathbb{C}^2)^{\otimes N}$. As $|W_N\rangle^{\otimes n}$ will be a linear combination of Dicke states, we can obtain an upper bound for $\mathrm{rk}(W_N^{\otimes n})$ by adding up the tensor ranks of each component Dicke state. Now each one of these corresponds exactly to a different way of separating n distinct excitations $|1\rangle$ into $k = 1, 2, \ldots, N$ local states. This number is equal to the Stirling number of the second kind, namely $S(n, k) = \frac{1}{k!}\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i} i^n$, where we only have to consider $k \leq n$ as any larger number of parties is taken care of by the symmetrization.

For example, $S(3, 2) = 3$ which implies that there are three ways of separating three excitations into two local systems, namely $|0^{\otimes 3}, \ldots, 0^{\otimes 3}, 100, 011\rangle$, $|0^{\otimes 3}, \ldots, 0^{\otimes 3}, 010, 101\rangle$, and $|0^{\otimes 3}, \ldots, 0^{\otimes 3}, 001, 110\rangle$. By permuting the local states, each of these generates a Dicke state with corresponding monomials $x_0^{N-2} x_4 x_3$, $x_0^{N-2} x_2 x_5$ and $x_0^{N-2} x_1 x_6$, respectively.

Since each of the $S(n, k)$ monomials representing the same separation $(n \to k)$ are related by a simple change in variables, each will have the same polynomial rank. Then by adding up all separations we obtain $\mathrm{rk}(W_N^{\otimes n}) \leq \sum_{k=1}^{\min\{N,n\}} S(n, k)\mathrm{pr}(x_0^{N-k} x_1 \cdots x_k) \leq \sum_{k=1}^{\min\{N,n\}} S(n, k) \times (1 + \max\{N - k, k\})2^{k-1}$, where the second inequality follows from [17], Cor. 4.5 and Prop. 11.6. In particular, this bound is of the form $f(n)N + g(n)$ with some functions $f(n)$ and $g(n)$. In other words,

Theorem 5.—$\mathrm{rk}(W_N^{\otimes n})$ is upper bounded by a linear function in N. Thus for large N, $|W_N\rangle^{\otimes n}$ can be prepared by LOCC from a GHZ-type state of rank linear in N. □

The large-n behavior of this bound is not very good, but based on a simple asymptotic consideration of the Stirling numbers for $n \approx \log N$, we find that

Corollary 6.—$\lim_{n\to\infty}\sqrt[n]{\mathrm{rk}(W_N^{\otimes n})} \leq O(\log N)$. □

Lower bound on the tensor rank of $|W_N\rangle^{\otimes n}$.

Lemma 7.—Any state of the form $|\Omega\rangle = |W_{N-1}\rangle^{\otimes n} + \sum_{k=1}^{n}\sum_{\pi\in S_N} c_{\pi k} U_\pi(|W_{N-1}\rangle^{\otimes k}|0_{N-1}\rangle^{\otimes(n-k)})$ is SLOCC equivalent to $|W_{N-1}\rangle^{\otimes n}$.

Proof.—We perform successively invertible SLOCC transformations on $|\Omega\rangle$, each transformation eliminating a term in the double sum. For instance, applying the transformation $|W_{N-1}\rangle \to |W_{N-1}\rangle - c_{\pi k}|0_{N-1}\rangle$, $|0_{N-1}\rangle \to |0_{N-1}\rangle$ on $|\Omega\rangle$ by local invertible operators will eliminate the term $U_\pi(|W_{N-1}\rangle^{\otimes k}|0_{N-1}\rangle^{\otimes(n-k)})$. The procedure is repeated on all terms in the sum until just $|W_{N-1}\rangle^{\otimes n}$ remains. □

To prove a lower bound, note that $\mathrm{rk}(W_N^{\otimes n})$ is the minimum number of product states whose linear span contains the set $S = \{|W_{N-1}\rangle, |0_{N-1}\rangle\}^{\otimes n}$. Each of these product states can be substituted with an element from $S\setminus\{|W_{N-1}\rangle^{\otimes n}\}$ to yield a new set whose linear span also contains S. Thus, $|W_{N-1}\rangle^{\otimes n}$ is a linear combination of elements from $S\setminus\{|W_{N-1}\rangle^{\otimes n}\}$ and at most $\mathrm{rk}(W_N^{\otimes n}) - (2^n - 1)$ product states. Thus by Lemma 7 we get $\mathrm{rk}(W_{N-1}^{\otimes n}) \leq \mathrm{rk}(W_N^{\otimes n}) - (2^n - 1)$. As proven in [18], for $N = 3$, $2^{n+1} - 1 \leq \mathrm{rk}(W_3^{\otimes n})$. From these two inequalities, a simple inductive argument provides part (a) in the next theorem; part (b) then immediately follows after observing that Theorem 5 reads $\mathrm{rk}(|W_N\rangle^{\otimes 2}) \leq 3N - 2$ when $n = 2$.

Theorem 8.—(a) $\mathrm{rk}(W_N^{\otimes n}) \geq (N-1)2^n - N + 2$, (b) $\mathrm{rk}(W_N^{\otimes 2}) = 3N - 2$. □

Multicopy and catalytic SLOCC transformations.—We now move on to the topic of SLOCC catalysis for multipartite entanglement transformations. Let $\mathcal{H} = \bigotimes_{k=1}^{n}\mathcal{H}_k$ and $\mathcal{H}' = \bigotimes_{k=1}^{n}\mathcal{H}'_k$ be n-partite quantum systems, and consider $\mathcal{H}_k$ and $\mathcal{H}'_k$ to be orthogonal to each other. Let $|\psi_0\rangle$ and $|\psi_1\rangle$ be two vectors from $\mathcal{H}$ and $\mathcal{H}'$, respectively. Then the direct sum of $|\psi_0\rangle$ and $|\phi_1\rangle$ is given by $|\psi_0\rangle \oplus |\psi_1\rangle \in \mathcal{H} \oplus \mathcal{H}' \subseteq \bigotimes_{k=1}^{n}(\mathcal{H}_k \oplus \mathcal{H}'_k)$. Notice that when $|\phi_1\rangle = \otimes_{k=1}^{n} L_k|\psi_1\rangle$ and $|\phi_2\rangle = \otimes_{k=1}^{n} L'_k|\psi_2\rangle$, we simply have $|\phi_1\rangle \oplus |\phi_2\rangle = \bigotimes_{k=1}^{n}(L_k \oplus L'_k)(|\psi_1\rangle \oplus |\psi_2\rangle)$. By induction one can immediately show that the SLOCC ordering is preserved under direct sums.

Lemma 9.—If $|\psi_k\rangle$ can be transformed into $|\phi_k\rangle$ via SLOCC, then $\bigoplus_k|\psi_k\rangle$ can also be transformed into $\bigoplus_k|\phi_k\rangle$ via SLOCC. □

We can use Lemma 9 to get a general relation between SMLOCC and SELOCC. Assume that $|\psi\rangle^{\otimes n}$ can be transformed into $|\phi\rangle^{\otimes n}$ via SLOCC for some $n \geq 1$. Then by choosing $|c\rangle = \bigoplus_{k=1}^{n}|\psi\rangle^{\otimes n-k} \otimes |\phi\rangle^{\otimes k}$, the result that $|\psi\rangle \otimes |c\rangle$ can be transformed to $|\phi\rangle \otimes |c\rangle$ via SLOCC follows from Lemma 9. So we get, similar to [19,20]:

Theorem 10.—If $|\psi\rangle$ can be transformed to $|\phi\rangle$ via SMLOCC, then the same transformation can also be achieved via SELOCC. □

By observation 1, to demonstrate the effect of entanglement catalysis, we only need to find a state $|\phi\rangle$ with the following property: $\mathrm{rk}(\phi) = n$ and there is some $k \geq 1$ such that $\mathrm{rk}(\phi^{\otimes k}) \leq (n-1)^k$. The source state $|\psi\rangle$ can be chosen as an n-partite GHZ state with tensor rank $(n-1)$. Such states $|\phi\rangle$ do exist as proven in the previous section.

In the following we shall provide two different constructions. The first class is given by the famous tripartite matrix multiplication tensor and the second one is given by the W_N states. By Theorem 10 these also suffice to show the existence of SELOCC transformations when the uncatalyzed transformation is impossible.

Theorem 11.—Let $|\Phi^{(3)}\rangle = |\Phi_2\rangle_{AB} \otimes |\Phi_2\rangle_{BC} \otimes |\Phi_2\rangle_{CA}$, where $|\Phi_2\rangle = |00\rangle + |11\rangle$, and let $|\psi\rangle_{ABC}$ be any generalized GHZ-type state with tensor rank 6. Then the transformation of $|\psi\rangle$ to $|\Phi^{(3)}\rangle$ cannot be realized by SLOCC but can be realized by both SMLOCC and SELOCC.

Proof.—It has been shown that $|\Phi^{(3)}\rangle$ is just the 2×2 matrix multiplication tensor [14,21]. By a well known result in algebraic complexity theory, $\mathrm{rk}(\Phi^{(3)}) = 7 > 6$ [22]. Hence, $|\psi\rangle$ cannot be SLOCC transformed into $|\Phi^{(3)}\rangle$. The best known algorithm for $d \times d$ matrix multiplication requires $O(d^{2.376})$ multiplication steps [23]. Hence the tensor rank of $|\Phi^{(3)}\rangle^{\otimes n}$, which corresponds to the algebraic complexity of $2^n \times 2^n$ matrix multiplication, is $O(2^{2.376n})$. On the other hand, the tensor rank of $|\Psi\rangle^{\otimes n}$ is simply $6^n = 2^{(\log_2 6)n} \approx 2^{2.585n}$, which is larger than $O(2^{2.376n})$ for sufficiently large n. Thus we have confirmed the existence of n (perhaps very large) such that $\mathrm{rk}(\Psi^{\otimes n}) \geq \mathrm{rk}((\Phi^{(3)})^{\otimes n})$. Both SMLOCC and SELOCC are possible.□

If we consider multipartite rather than tripartite state spaces, the W states provide much simpler examples.

Theorem 12.—For any $N \geq 5$ the transformation of $|\mathrm{GHZ}_N^{N-1}\rangle$ to $|W_N\rangle$ cannot be realized by SLOCC but can be achieved by both SMLOCC and SELOCC. Furthermore, two copies are sufficient in SMLOCC, and the catalyst in SELOCC can be chosen as $|W_N\rangle \oplus |\mathrm{GHZ}_N^{N-1}\rangle$.

Proof.—The result follows immediately from the facts that $\mathrm{rk}(W_N^{\otimes 2}) = 3N - 2$ in Theorem 8 and $(N-1)^2 \geq 3N - 2$ for $N \geq 5$. One can easily see that the rank of the GHZ state can indeed be chosen as $\lceil\sqrt{3N-2}\rceil$, which is much smaller than $N - 1$ for $N \gg 1$. □

Conclusions.—We have shown that the theory of homogeneous polynomials can be used to obtain insights on the symmetric tensor rank of symmetric states. Via this connection, we proved upper and lower bounds on the tensor rank for one and multiple copies of W_N states as well as the exact tensor and symmetric tensor rank of multiqubit Dicke states. We then proceeded to show that multicopy and catalytic activation of otherwise impossible SLOCC transformations exists, using our results on W_N states to find explicit low-dimensional examples.

Our work suggests several open questions which we leave for future investigation. First, given two states $|\psi\rangle$ and $|\phi\rangle$, what are the necessary and sufficient conditions such that $|\psi\rangle$ can be converted to $|\phi\rangle$ under SMLOCC and SELOCC? When $|\psi\rangle$ is a generalized GHZ state, the question becomes completely a matter of tensor rank multiplicativity. Asked in a different way, for some target state $|\phi\rangle$, when does there exist a state $|\psi\rangle$ such that transformation $|\psi\rangle$ to $|\phi\rangle$ is possible under SMLOCC and SELOCC but impossible with just single copies. Another relevant problem is to determine the asymptotic tensor rank of $|W_3\rangle$, and more generally of $|W_N\rangle$. Note that our lower bound of 2 coincides with the *border rank* [24]. It is conceivable that the asymptotic rank is 2 for all N, but even an improvement of our logarithmic upper bound would be interesting.

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