Entanglement cannot make imperfect quantum channels perfect

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Entangled inputs can enhance the capacity of quantum channels, this being one of the consequences of the celebrated result showing the non-additivity of several quantities relevant for quantum information science. In this work, we answer the converse question–whether entangled inputs can ever render noisy quantum channels have maximum capacity–to the negative: No sophisticated entangled input of any quantum channel can ever enhance the capacity to the maximum possible value; a result that holds true for all channels both for the classical as well as the quantum capacity. This result can hence be seen as a bound as to how "non-additive quantum information can be". We find several practical and remarkably simple computable single-shot bounds to capacities, related to entanglement measures. As examples, we discuss the qubit amplitude damping and identify the first meaningful bound for its classical capacity.

How much information can one transmit reliably through a quantum channel such as a telecommunication fiber? This basic question is—despite much progress in recent years [1–5]—still surprisingly wide open. Some suitable encoding and decoding is necessary, needless to say, but the optimal achievable rates can still not be expressed in a computable closed form. For classical information, the hope that the single-shot capacity would be sufficient to arrive at that goal—corroborated by many examples of channels for which this is in fact true [2]—was found to be unjustified with the celebrated result [1] on the non-addivity of several quantities that are in the center of interest in quantum information science [3–5]. In particular, entangled inputs help and do increase the classical information capacity. This result showed that the question of finding capacities of quantum channels is more complicated than what one might have antecipated. In the case of quantum information transmission, a similar situation has been known to be true already for a long time: in general one must regularize the single-shot expression, given by the coherent information, in order to attain the quantum capacity [6].

To contribute to fixing the coordinate system of channel capacities, this insight begs for a resolution of the following question: To what extent can entanglement help then? Is the mentioned result rather an academic observation, manifesting itself in small violations of additivity in high physical dimensions? An interesting question in this context is the following: *Can entanglement render noisy quantum channels take their maximum possible capacity or make them even perfect, if only suitably entangled inputs are allowed for?* This would be the other extreme, where the non-additivity serves as a resource to overcome the noisiness of channels.

In this work, we answer this question to the negative: For all quantum channels, no matter how elaborate the entangled coding over many uses of the channel might be, one can never achieve the maximum possible capacity if this is not already true on the single-shot level. This observation holds true both for the classical as well as the quantum capacity.

We show this by introducing new upper bounds to these capacities, which can be evaluated on the single-shot level and

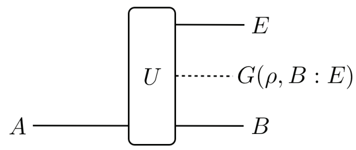

FIG. 1. Upper bound to the classical information capacity in terms of entanglement measures between the output and its environment in a dilation of the quantum channel.

are computable. We connect questions of capacities to those of entanglement measures of systems and their environments. These bounds are useful in their own right, which will be shown by means of an example of amplitude-damping channel.

Notation and setting. We start our discussion by fixing the notation and clarifying some basic concepts that will be used later on. We consider general quantum channels of arbitrary finite dimension, $T : \mathcal{S}(\mathbb{C}^d) \to \mathcal{S}(\mathbb{C}^d)$, modelling any general noisy quantum evolution. T is hence an arbitrary trace-preserving completely positive map. Such a channel can always be written in terms of a *Stinespring dilation* as

$$T(\rho) = \text{tr}_E(U\rho U^\dagger), \tag{1}$$

labelling the input by A, associated with the Hilbert space $\mathbb{C}^d$, the output by B and the environment by E, equipped with Hilbert spaces $\mathbb{C}^d$ and $\mathbb{C}^{d_{env}}$, respectively. U is an isometry mapping the input on A onto a quantum state on B and E.

The classical information capacity, or short *classical capacity*, of a quantum channel is the rate at which one can reliably send classical information. It is related to the *Holevo-*χ [10] or the *single-shot classical capacity* of that channel,

$$\chi(T) = \max\Big(S\big(\sum_j p_j T(\rho_j)\big) - \sum_j p_j S(T(\rho_j))\Big), \tag{2}$$

where the maximum is taken over probability distributions and states, as the asymptotic regularization

$$C(T) = \limsup_{n\to\infty} \frac{\chi(T^{\otimes n})}{n}. \tag{3}$$

The trivial maximum value of the capacity is given by the *maximum output entropy* of the quantum channel,

$$C(T) \leq \max_{\rho} S(T(\rho)) = S_{\max}(T). \tag{4}$$

We will say that whenever this bound is saturated, so when $C(T) = S_{\max}(T)$, that the channel has *maximum capacity*, giving rise to the maximum that is trivially possible. Of course, this notion includes the situation of a *perfect quantum channel* that has maximum output entropy of $C(T) = S_{\max}(T) = \log_2(d)$.

The *quantum capacity* of a quantum channel, in turn, is related to the rate at which one can reliably send quantum information through a quantum channel. Writing

$$Q_1(T) = \max_{|\psi\rangle} \left(S(\omega_B) - S(\omega_E)\right), \tag{5}$$

calculated in the state $\omega = |\phi\rangle\langle\phi|$ as a state on C, B, E, and where $|\phi\rangle := U|\psi\rangle$, with U being again the isometry of T, mapping A to B and E, and $|\psi\rangle$ being a state vector on C and A. The quantum capacity is then

$$Q(T) = \limsup_{n\to\infty} \frac{Q_1(T^{\otimes n})}{n}, \tag{6}$$

again, referred to as *maximum* if $Q = S_{\max}(T)$.

Main result. We can now formulate the main result.

Observation 1 (Entanglement cannot enhance classical capacity of noisy quantum channels to its maximum value)**.** *Every quantum channel that is noisy—in the sense that the single-shot classical capacity is not the maximum output entropy—cannot be made having maximum capacity under the help of any sophisticated entangled input.*

So if there is a gap to the maximum possible single-shot capacity, this gap will be preserved in the asymptotic limit, independent of n: No entangled input can overcome this limitation. The single-shot classical capacity may be non-additive, as has been shown in Ref. [1]. Yet, entanglement can only help to some extent, and can, in particular, not make any imperfect channel perfect.

Upper bounds for classical capacities. In order to show this result and the equivalent one for the quantum capacity, we make use of upper bounds to channel capacities, starting with the classical capacity. The bounds forming the tools of the argument will be provided by quantities that capture the entanglement between a system and its environment in a dilation of the channel. We first show what properties a general quantity $M : \mathcal{S}(\mathbb{C}^d \otimes \mathbb{C}^{d_{env}}) \to \mathbb{R}^+$, defined on bipartite quantum systems, should have. In order to be entirely clear, we will always give the tensor factors with respect to which an entanglement measure will be taken. For example, $M(\sigma, A : B)$ would be the quantity evaluated for σ with respect to the split $A : B$. Two properties will be important:

1. M has the property that

$$E_F(\sigma, A : B) \geq \sum_{j=1}^{n} M(\sigma_{A_j,B_j}, A_j : B_j) \tag{7}$$

for every bipartite state σ defined on n copies of a $d\times e$-dimensional quantum system, labeled $B_1, \ldots, B_n$ and $A_1, \ldots, A_n$, σ_{B_j,A_j} denoting the respective reduction.

2. M is *faithful*. That is, $M(\rho, A : B) > 0$ for bipartite states ρ on A and B if and only if ρ is entangled with respect to this split.

Here, E_F denotes the *entanglement of formation* [3]. As it turns out, for any quantity satisfying Property 1, the following bound holds true:

Observation 2 (Upper bound for the classical capacity)**.** *For any quantum channel T and any quantity M that satisfies the condition 1. we find the single-shot upper bound*

$$C(T) \leq \max_{\rho} \left(S(T(\rho)) - M(U\rho U^\dagger, B : E)\right). \tag{8}$$

The argument leading to this bound is remarkably simple: Starting from Eq. (2), and defining $\sigma = U^{\otimes n}\rho(U^{\otimes n})^\dagger$ with reductions $\sigma_{B_1,\ldots,B_n} = \text{tr}_{E_1,\ldots,E_n}(\sigma)$, A being formed by $A_1, \ldots, A_n$ and B by $B_1, \ldots, B_n$, we find, using the MSW-correspondence [9],

$$\begin{aligned}\chi(T^{\otimes n}) &= \max_{\rho} \left(S(T^{\otimes n}(\rho)) - E_F(U^{\otimes n}\rho(U^{\otimes n})^\dagger, B : E)\right) \\ &= \max_{\rho}\Big(S(\sigma_{B_1,\ldots,B_n}) - E_F(\sigma, B : E)\Big) \\ &\leq \max_{\rho}\Big(\sum_{j=1}^{n} S(\sigma_{B_j}) - E_F(\sigma, B : E)\Big),\end{aligned} \tag{9}$$

using subadditivity, and hence, using Property 1,

$$\begin{aligned}\chi(T^{\otimes n}) &\leq \max_{\rho}\sum_{j=1}^{n}\Big(S(\sigma_{B_j}) - M(\sigma_{B_j,E_j}, B_j : E_j)\Big) \\ &\leq n \max_{\rho} \left(S(T(\rho)) - M(U\rho U^\dagger, B : E)\right),\end{aligned} \tag{10}$$

which is the above single-shot bound of Observation 2.

This bound is to be compared with the MSW expression [9] for the Holevo-χ itself,

$$\chi(L) = \max_{\rho}(S(T(\rho)) - E_F(U\rho U^\dagger, B : E)). \tag{11}$$

This is very similar, except that now the entanglement of formation takes the role of the quantity M. This indeed leads also to the conclusion of Observation 1 for the classical capacity: $C(L)$ achieves the maximum upper bound $\max_\rho S(T(\rho))$ if and only if $\chi(L)$ achieves it. This is because χ achieves it if and only if

$$E_F(U\rho U^\dagger, B : E) = 0 \tag{12}$$

for the maximizing ρ in $\max_\rho S(T(\rho))$, which means that $U\rho U^\dagger$ has to be separable. Now, if M is also faithful, i.e., it satisfies Property 2, then we can see that also C achieves $\max_\rho S(T(\rho))$ iff the optimal $U\rho U^\dagger$ is separable [12], which proves Observation 1. Below we shall provide a list of quantitites, most of them satisfying both of the postulates.

Identifying candidates for suitable entanglement measures. This result, needless to say, leaves the question of finding entanglement measures exhibiting the above properties 1. and 2. I.e. we need at least one such measure to prove the claim. Moreover, any computable measure satisfying 2. will give rise to a useful bound for capacity.

(a) The entanglement measure G: Define as in Ref. [7]

$$C_{\leftarrow}(\rho, B:E) = S(\rho_B) - \inf \sum_{j=0}^{k-1} q_j S\left(\frac{\mathrm{tr}_E((\mathbb{1}\otimes P_j)\rho(\mathbb{1}\otimes P_j)^\dagger)}{q_j} \right), \quad (13)$$

where the infimum is performed over all Kraus operators $P_0, \ldots, P_{k-1}$, $\sum_{j=0}^{k-1} P_j^\dagger P_j = \mathbb{1}$, and $q_j = \mathrm{tr}((\mathbb{1}\otimes P_j)\rho(\mathbb{1}\otimes P_j)^\dagger)$. This is a computable single-shot quantity. We denote the convex hull of this function with G,

$$G(\rho, B:E) = \min \sum_j p_j C_{\leftarrow}(\rho_j, B:E), \quad (14)$$

where $\rho = \sum_j p_j \rho_j$, and which is an "entanglement measure" in its own right (it is at least a monotone under one-local LOCC). We claim that this function has the right properties.

Observation 3 (Bounding capacities in terms of classical correlations)**.** *The quantity G has the properties 1 and 2.*

In fact, the validity of Property 2 is easily shown: Every separable state will have a convex combination in terms of products, for each of which $C_{\leftarrow}$ will vanish. In turn, if a state is entangled, then there must in any convex combination be at least an entangled and hence correlated term, which will be detected by $C_{\leftarrow}$. To show Property 1, we can make use of a result of Ref. [13]: For a pure tripartite state ρ shared by A, B, and C, a duality relation gives rise to

$$S(\rho_A) = E_F(\rho_{A,B}, A:B) + C_{\leftarrow}(\rho_{A,C}, A:C). \quad (15)$$

Using the step of Refs. [11, 13] iteratively, one therefore finds

$$\begin{aligned} E_F(\rho, AB:CD) &= \sum_j p_j S(\rho_{j,A,B}) \\ &\geq \sum_j p_j (E_F(\rho_{j;A,C}, A:C) + C_{\leftarrow}(\rho_{j;B,D}, B:D) \\ &\geq E_F(\rho_{A,C}, A:C) + G(\rho_{B,D}, B:D), \end{aligned} \quad (16)$$

arriving at Property 1. This gives rise to a computable bound. Explicitly, it reads

$$C(T) \leq \max_{(\{p_j\},\{\rho_j\})} \left(S(T(\rho)) - \sum_j p_j C_{\leftarrow}(U\rho_j U^\dagger, B:E) \right), \quad (17)$$

with $\rho = \sum_j p_j \rho_j$, as a single maximization. A lower bound to this is

$$C(T) \leq \max_\rho S(T(\rho)) - \min_\rho C_{\leftarrow}(U\rho U^\dagger, B:E), \quad (18)$$

which is usually less tight, but much simpler to compute.

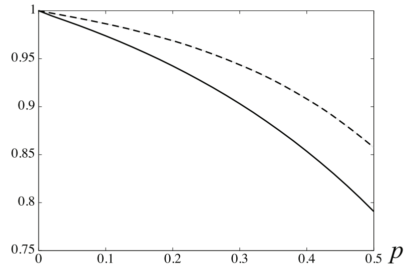

FIG. 2. Upper bound to the classical information capacity of the qubit amplitude damping channel as a function of $p \in [0, 1/2]$. The chosen Kraus operators delivering good bounds for $k = 3$ are given by $P_0 = |0\rangle\langle 0|/2$, $P_1 = \sum_{j,k=0,1} |j\rangle\langle k|/x$, $P_2 = (\mathbb{1} - P_0^\dagger P_0 - P_1^\dagger P_1)^{1/2}$ for $x = 4$ (dashed) and $x = 3$ (solid).

(b) Variants of the relative entropy of entanglement: The measure proposed in Ref. [14] is *superadditive* and not larger than the entanglement of formation, implying Property 1. It is also shown to be faithful in Ref. [14], which is Property 2.

(c) Squashed entanglement: The *squashed entanglement* [15] E_{sq} is also known to be superadditive and is bounded from above by the entanglement of formation, so qualifies as a bound for the same reason. It is not easily computable, however, as it is based on a construction involving a state extension the dimension of which is not bounded. However a lower bound to squashed entanglement was provided in Ref. [17]:

$$E_{\mathrm{sq}}(U\rho U^\dagger, B:E) \geq \frac{1}{4\ln(2) d d_{env}} \left(\min_\sigma \|U\rho U^\dagger - \sigma\|_1 \right)^2, \quad (19)$$

in terms of the trace-norm distance to the set of separable quantum states σ with respect to the split $B:E$.

(d) Distillable entanglement: A not efficiently computable but in instances practical bound is provided by the LOCC or PPT *distillable entanglement* with respect to $B:E$. (Note that either version of distillable entanglement does not satisfy property 2.)

Example: The amplitude qubit damping channel. To find any non-trivial bound for the capacity of the amplitude damping channel has been an open problem for some time [18]. The methods proposed here give rise to such bounds. The Kraus operators of $T(\rho) = K_0 \rho K_0^\dagger + K_1 \rho K_1^\dagger$ are given by

$$K_0 = \sqrt{p}|0\rangle\langle 1|, \quad K_1 = |0\rangle\langle 0| + \sqrt{1-p}|1\rangle\langle 1| \quad (20)$$

for $p \in [0, 1]$. The isometry U of this qubit channel maps

$$|0\rangle \mapsto |0,0\rangle, \quad |1\rangle \mapsto \sqrt{p}|0,1\rangle + \sqrt{1-p}|1,0\rangle. \quad (21)$$

To bound the correlation measure $C_{\leftarrow}(U\rho U^\dagger, B:E)$, any choice for k and for $P_0, \ldots, P_{k-1}$ giving rise to a positive operator valued measure amounts to a valid bound. This gives rise to the bound depicted in Fig. 2 [20] for $p \in [0, 1/2]$. Note that it is *significantly* tighter than the trivial bound $C(T) \leq S_{\max}(T)$, which here takes the value 1. It is easy to see that for $p \in [0, 1/2]$ there always exists an input diagonal in the computational basis that yields an output $\mathrm{diag}(1, 1)/2$ with unit entropy. For $p = 0$ the channel becomes the perfect channel with $C(T) = 1$. (The *entanglement assisted classical information capacity* [16] is also a crude upper bound, but yields values even larger than 1 for $p \in [0, 1/2]$). We have hence established a first non-trivial bound for the amplitude damping channel. Needless to say, the same techniques can be applied to any finite-dimensional quantum channel.

Quantum capacity. Indeed, an argument very similar to the above one for the classical capacity of a quantum channel holds true also for the quantum capacity. Now $S(\omega_E)$ in Eq. (5) is just the entropy of entanglement of ω in the partition $AB : E$. We can then proceed just as in the classical capacity case and get the upper bound

$$Q(T) \leq \max_{\rho} \big(S(T(\rho_A)) - M(U\rho U^\dagger) \big) \tag{22}$$

of states on systems C and A, for a measure M with the same properties as in the the classical capacity case (now calculated with respect to the split $CB : E$). For the bound on the right hand side to be maximal, there must exist a ρ such that $U\rho U^\dagger$ is separable (since we chose a faithful quantity M) and

$$S(T(\omega_A)) = S_{\max}(T). \tag{23}$$

If this is the case, there must exist a convex decomposition of $\rho = \sum_j p_j |\psi_j\rangle\langle\psi_j|$ (always finite, by virtue of Caratheodory's theorem) such that for all j, the state vector $U|\psi_j\rangle$ is a product in the cut $CB : E$. Thus, from the concavity of the entropy, we find that for at least one of the $|\psi_j\rangle$

$$\begin{aligned} &S(T(|\psi_j\rangle\langle\psi_j|)) - S(\mathrm{tr}_{C,B}(U|\psi_j\rangle\langle\psi_j|U^\dagger)) \\ &= S(T(|\psi_j\rangle\langle\psi_j|)) = S_{\max}(T). \end{aligned} \tag{24}$$

Therefore,

$$Q_1(T) \geq S(T(|\psi_j\rangle\langle\psi_j|)) - S(\mathrm{tr}_{C,B}(U|\psi_j\rangle\langle\psi_j|U^\dagger)) = S_{\max}(T), \tag{25}$$

and we find that $Q_1(T)$ must be maximal too. Hence, we arrive at the following conclusion. So again, entanglement can help to a certain degree, but never uplift channels to the maximum possible value. Note finally that Eq. (22) constitutes the best known computable upper bound to the quantum capacity of a channel.

Observation 4 (Entanglement cannot enhance the quantum capacity to its maximum value)**.** *For every quantum channel for which the single-shot quantum capacity is not yet already given by the trivial upper bound $S_{\max}(T)$, the same will hold true for the quantum capacity.*

Summary and outlook. In this work, we have investigated the converse question to the additivity problem: How much can entanglement help enhance capacities of quantum channels. In the focus of interest was the question whether entanglement can ever enhance the capacity to its trivial maximum if a single invocation does not yet reach that. We affirmatively answer that question to the negative, including the quantum and classical capacity. In doing so, we have established practical computable upper bounds to capacities, relating them to entanglement measures and rendering bounds and witnesses to the latter quantities useful to assess capacities. There is though an interesting challenge: all the quantities from our list exhibit a sort of monogamy, i.e., for states which are highly sharable they have to be small, implying that the bounds may become loose [23]. An open question is therefore how to find a quantity which would satisfy our Property 1, but would not necessarily drop for sharable states. It is the hope that the present work triggers further work on how "small" violations of additivity really are in practice and what role entanglement plays after all in quantum communication.

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$$C_{\leftarrow}(\rho, B : E) = \sum_{j=0}^{k-1} q_j S(\rho_B^{(j)} || \rho_B). \tag{26}$$

Still, the quantity $C_{\leftarrow}$ is not convex.
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